Uniform Asymptotic Approximations of Integrals

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. Further, I declare that the work has not been submitted for any other degree or professional qualification except as specified.

(Sarah Farid Khwaja),

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The candidate confirms that the work submitted is his own, except where work which has formed part of jointly-authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

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This thesis includes some of the material that also appears in the following jointly authored papers. The author has done main part of the work.

- Khwaja, Sarah Farid; Olde Daalhuis, Adri B., *Uniform asymptotic expansions for hypergeometric functions with large parameters IV*. Accepted for the Frank Olver special issue of Analysis and Applications, March 2014.
- Khwaja, Sarah Farid; Olde Daalhuis, Adri B. *Exponentially accurate uniform asymptotic approximations for integrals and Bleistein's method revisited.* Proc. R. Soc. A, 8 May 2013, Vol. 469, No. 2153.
- Khwaja, S. Farid; Olde Daalhuis, A. B. *Uniform asymptotic approximations for the Meixner-Sobolev polynomials.* Analysis and Applications, July 2012, Vol. 10, No. 03 : pp. $345 - 361$.

To my lovely father Prof. Dr. Farid Khwaja
and
To the memory of my mother Raana Farid,
who helped me to become the person I am.

Abstract

In this thesis uniform asymptotic approximations of integrals are discussed. In order to derive these approximations, two well-known methods are used *i.e.*, the saddle point method and the Bleistein method. To start with this, examples are given to demonstrate these two methods and a general idea of how to approach these techniques.

The asymptotics of the hypergeometric functions with large parameters are discussed *i.e.*, ${}_2F_1\left(\begin{matrix} a+e_1\lambda, b+e_2\lambda \\ c+e_3\lambda \end{matrix}; z\right)$ where $e_j = 0, \pm 1$, $j = 1, 2, 3$ as $|\lambda| \to \infty$, which are valid in large regions of the complex z-plane, where a, b and c are fixed. The saddle point method is applied where the saddle point gives a dominant contributions to the integral representations of the hypergeometric functions and Bleistein's method is adopted to obtain the uniform asymptotic approximations of some cases where the coalescence takes place between the critical points of the integrals.

As a special case, the uniform asymptotic approximation of the hypergeometric function where the third parameter is large, is obtained. A new method to estimate the remainder term in the Bleistein method is proposed which is created to deal with new type of integrals in which the usual methods for the remainder estimates fail.

Finally, using the asymptotic property of the hypergeometric function when the third parameter is large, the uniform asymptotic approximation of the monic Meixner Sobolev polynomials $S_n(x)$ as $n \to \infty$, is obtained in terms of Airy functions. The asymptotic approximations for the location of the zeros of these polynomials are also discussed. As a limit case, a new asymptotic approximation for the large zeros of the classical Meixner polynomials is provided.

Lay Summary

The focus of this thesis is on hypergeometric functions, which are special functions and solution of a specific second order linear differential equation. We express these hypergeometric functions in terms of their integral representations.

Over the years researchers have worked upon some approximations of these interesting integrals, given one of the parameters in the hypergeometric function is large. However there was no unified analysis of all the cases of these integrals.

In this thesis, a unified analysis and discussion of all the cases is available. To derive all the cases, two well-known approximations namely the Bleistein method and the saddle point method are used. Moreover application of these two methods is illustrated to find the uniform asymptotic approximation of monic Meixner-Sobolev polynomials.

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Chapter 1

Introduction

In Mathematics, we often use approximations when an exact quantity is unknown or difficult to obtain. In order to obtain these approximations, we strive to develop accurate and concise estimates of quantities of interest. The approximations are obtained in terms of constants and standard functions, which simplifies the process of analysis. Asymptotic approximations are of paramount importance in applied analysis and computational Mathematics. In general, asymptotic approximations tend to describe the limiting behaviour of a function. The methodology has applications in many fields in Science and Engineering, such as examining the behaviour of physical systems in fluid mechanics when they are large. It also has wide applications in statistics and probability. Even though asymptotic approximations is an old subject introduced by Laplace, new methods and techniques continue to appear in the literature.

In this thesis, we deal with the uniform asymptotic approximations in the analysis of special functions using integrals. This thesis consists of two published papers [20], [21] and one paper which is near completion.

In this chapter, we define the relevant definitions in the area of asymptotics. Also a brief discussion about the saddle point method and the Bleistein method is given. These methods are illustrated with the help of examples. Moreover, a literature review of asymptotic approximations of Gauss hypergeometric functions is also given. An effort is made to keep the introductory material brief. Adequate references are given so that the reader can refer to them for further reading. This chapter concludes with a discussion of layout and contribution of this thesis.

The focus of this thesis is on hypergeometric functions, which are special functions and solution of a specific second order linear differential equation. We express these hypergeometric functions in terms of their integral representations. Over the years, researchers have worked upon some approximations of these interesting integrals. However there was no unified analysis of all the cases of these integrals. In this thesis, a unified analysis and discussion of all the cases is available. To derive all the cases, two well-known methods namely the saddle point method and Bleistein's method are used. Moreover application of these two methods is illustrated by obtaining the uniform asymptotic approximation of monic Meixner-Sobolev polynomials.

1.1 Asymptotics

Asymptotics refers to the branch of mathematics which investigate a function's behaviour, when one or more than one parameter in the function becomes arbitrarly large. Asymptotic approximations/expansions are formal truncated series which provide an approximation to a given function as the argument tends to a particular point. The power series is of the most common example of an asymptotic expansion such as

$$e^z \sim 1 + z + \frac{z^2}{2!} + \mathcal{O}(z^3), \qquad z \to 0. \tag{1.1}$$

This subject is mainly divided into two main areas, *i.e.* integrals and differential equations. Asymptotic analysis is a key tool for exploring ordinary differential equations which arises in mathematical models of real problems. It is also an important method to investigate the behaviour of integrals such as Laplace type integrals as one of the parameters in the integrand tend to infinity. These two subfields of asymptotic analysis are quite rich in literature and have emerged as separate fields of study.

The area of asymptotic analysis was developed mainly in the early 20th century when researchers from different areas of expertise started looking for solution techniques to solve differential equations and integral equations. Poincaré and Stieltjes used the concept of asymptotic analysis in describing the important properties of asymptotic series. It is interesting to note that the first dedicated book on this subject was published in 1956 by Erdelyi [10]. The author explained the basic elements of asymptotic expansions and explained various methods of obtaining asymptotic approximations using integrals with large parameters and the solutions of ordinary linear differential equations. Later on De Brujin's book [7] was published in 1958 which was based on exotic results of asymptotic approximations. He covered most of the techniques of obtaining asymptotic approximations with the help of detailed examples. An extensive study of the saddle point method with worked examples and its applications can be found in this book. Frank Olver is well known for his contribution in the field of asymptotics. His book on Asymptotic and special functions [31] is a well-known book which is more oriented on the complete exposition of the asymptotic theory of differential equations and special functions. He also focused on the error analysis of the asymptotic expansions which were not considered before. Significant contributions in the field of asymptotics have been made by Wong [47] and Temme (see [42], [14], [39],[30], [43] and [40]) and most of their work is based on the uniform asymptotic approximations of integrals. In a recent paper by Temme [40], he discussed the basic methods of obtaining asymptotic expansions of integrals. He also gave an overview of the uniform asymptotic expansions of some of the special functions like Airy functions and Bessel functions with large parameters followed by examples. The research in asymptotic approximation is carried forward by many other mathematician like Olde Daalhuis ([25], [26], [28],[29] and [30]), Jones ([17], [19] and [18]), and López (see [22] and [23]). They have made significant contributions in the areas of asymptotic approximations for integrals and differential equations.

1.1.1 The $\mathcal{O}$, o and $\sim$ Symbols

To study the behaviour of an unknown function $f(x)$ in terms of a known function $\phi(x)$ defined in a domain $\mathcal{D} \in \mathbb{C}$ as $x \to \infty$, we explain the following concepts of asymptotic analysis.

Assuming $\phi(x) \neq 0$, $x \in \mathcal{D}$, if $|f(x)/\phi(x)|$ is bounded, then we write

$$f(x) = \mathcal{O}(\phi(x)), \qquad x \to \infty. \tag{1.2}$$

In other words we say that f is of order not exceeding ϕ.

Example 1.1. The following are a examples of the $\mathcal{O}$-symbol:

$$\begin{aligned} \sinh x &= \mathcal{O}(e^x), \qquad x \to \infty, \\ \frac{5\sin x}{x^2} &= \mathcal{O}\left(\frac{1}{x^2}\right), \qquad x \to \infty. \end{aligned} \tag{1.3}$$

If $f(x)/\phi(x) \to 0$, we express

$$f(x) = o(\phi(x)), \qquad x \to \infty. \tag{1.4}$$

In other words we say that f is of order less than ϕ.

Example 1.2. The following are the examples of the o-symbol:

$$\begin{aligned} e^{-x} &= o(1), \qquad x \to \infty. \\ \sqrt{x} &= o(x), \qquad x \to \infty. \end{aligned} \tag{1.5}$$

If $f(x)/\phi(x)$ tends to unity i.e.

$$\lim_{x\to\infty} \frac{f(x)}{\phi(x)} = 1, \tag{1.6}$$

we say that $f(x) \sim \phi(x)$, i.e. $f(x)$ is asymptotically equal to $\phi(x)$.

Example 1.3. The following are a few examples of $\sim$ symbol:

$$\begin{aligned} \sin x &\sim x, \qquad x \to 0, \\ n! &\sim e^{-n} n^n \sqrt{2\pi n}, \qquad n \to \infty. \end{aligned} \tag{1.7}$$

Definition 1.1 (Asymptotic sequence)**.** A sequence $\{\phi_n(x)\}$ for $n = 0, 1, 2, 3, \cdots$ is called an asymptotic sequence for large x if

$$\phi_{n+1}(x) = o(\phi_n(x)), \qquad \text{as} \quad x \to \infty. \tag{1.8}$$

Example 1.4. The following sequence is an example of an asymptotic sequence

$$1, \frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \cdots, \qquad \text{as} \quad x \to \infty. \tag{1.9}$$

Definition 1.2 (Asymptotic expansion)**.** Poincaré's definition states that if $\phi_n(z)$ is an asymptotic sequence as $z \to \infty$, then the series $\sum_{n=0}^{\infty} a_n \phi_n(z)$ (which may be divergent or convergent) is said to be an asymptotic expansion of a function $f(z)$, if for all $N > 0$ we have

$$f(z) = \sum_{n=0}^{N-1} a_n \phi_n(z) + \mathcal{O}(\phi_N(z)), \tag{1.10}$$

as $z \to \infty$, and is denoted by

$$f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z), \tag{1.11}$$

as $z \to \infty$.

1.2 Hypergeometric functions

In Mathematics and Physics, we encounter many special functions which are special cases of hypergeometric functions. These are the most important form of special functions which have been studied from various points of view. Gauss introduced hypergeometric functions and studied their properties. Later on, in the 19th century Riemann and Kummer made significant contributions to the theory of hypergeometric functions by studying analytic properties of the hypergeometric functions by means of differential equations.

Definition 1.3 (Gauss hypergeometric function)**.** The *Gauss hypergeometric function* is defined by the Gauss series

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{s=0}^{\infty} \frac{(a)_s\,(b)_s}{(c)_s\, s!} z^s, \tag{1.12}$$

on the disk $|z| < 1$ and by analytic continuation elsewhere.

In general ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)$ does not exists when $c = 0, -1, -2, \cdots$, but by defining it as

$$\mathbf{F}\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \frac{1}{\Gamma(c)} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{s=0}^{\infty} \frac{(a)_s\,(b)_s}{(c+s)_s\, s!} z^s, \tag{1.13}$$

it becomes an entire function for $a, b, c \in \mathbb{C}$ on the disk $|z| < 1$ and by analytic continuation elsewhere. (For detail we refer to [1], [28] and [41]).

The branch obtained by introducing a cut from 1 to ∞ on the real z-axis (*i.e.* the branch in the sector $|\mathrm{ph}\,(1-z)| \leqslant \pi$) is the principle branch of the hypergeometric function (1.12). As a multivalued function of z, ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)$ is analytic everywhere except for the possible branch points 0, 1 and ∞.

In (1.12), the term $(a)_n$ is known as *the Pochhammer symbol* or *the shifted factorial* and is defined as

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), \qquad (a)_0 = 1, \tag{1.14}$$

which can be written as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad a \neq 0, -1, -2, \cdots, \tag{1.15}$$

where Γ is the *gamma function* [4] and is defined by the following integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \qquad \Re(z) > 0. \tag{1.16}$$

In (1.12) if $a\, or\, b \in \mathbb{Z}^-$, then we obtain truncated series which is reducible to a polynomial. For example if $a = -m$, then we obtain

$${}_2F_1\left(\begin{matrix} -m, b \\ c \end{matrix}; z\right) = \sum_{s=0}^{m} \frac{(-m)_s\,(b)_s}{(c)_s} z^s = \sum_{s=0}^{m} (-1)^s \binom{m}{s} \frac{(b)_s}{(c)_s} z^s. \tag{1.17}$$

It is also possible to express the orthogonal polynomials like Jacobi and Legendre polynomials in terms of Gauss hypergeometric functions. For example, the Jacobi polynomials are expressed as

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-x}{2}\right), \tag{1.18}$$

and the Legendre polynomials can be expressed in terms of Gauss hypergeometric function

$$P_n(x) = {}_2F_1\left(\begin{matrix} -n, n+1 \\ 1 \end{matrix}; \frac{1-x}{2}\right). \tag{1.19}$$

More relations of the hypergeometric functions with other functions and orthogonal polynomials are given in Section 15.9 in [28].

The hypergeometric functions can be characterised in the following three main ways:

1. Functions represented by series expansion with coefficients satisfying the recursion properties; i.e. a hypergeometric series is a series $\sum C_n$ such that $\frac{C_{n+1}}{C_n}$ is a rational function and it satisfies
$$\frac{C_{n+1}}{C_n} = \frac{(n+a)(n+b)x}{(n+c)(n+1)}. \tag{1.20}$$

2. Functions being a solution of a second order linear differential equation *i.e.*,

$$z(1-z)\frac{d^2w}{dz^2} + (c-(a+b-1)z)\frac{dw}{dz} - abw = 0, \tag{1.21}$$

which was introduced by Euler and then studied by Gauss.

3. Functions that can be defined by integrals such as Mellin-Barne's integral, *i.e.*,

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};z\right) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(a+t)\Gamma(b+t)\Gamma(-t)}{\Gamma(c+t)}(-z)^t\, dt, \tag{1.22}$$

where $a, b \neq 0, 1, 2, \cdots$ and $|\mathrm{ph}\,(-z)| < \pi$.

1.2.1 Hypergeometric differential equation

The hypergeometric function defined in (1.12) satisfies the 2nd order linear differential equation (1.21) which is known as *the hypergeometric differential equation* (see Section 15.10 in [28]). Any 2nd order linear differential equation having three regular singularities can be converted in the form of a hypergeometric differential equation. This hypergeometric differential equation (1.21) was introduced by Euler in (1769). Later it was studied extensively by Gauss (1812), Kummer (1836) and Riemann (1857) (for details see [1])).

The regular singularities of this differential equation (1.21) are located at $z = 0, 1, \infty$, with exponents $0, 1-c; 0, c-a-b;$ and a, b respectively. In the case that $c, c-a-b, a-b \notin \mathbb{Z}$, then (1.21) will have two fundamental solutions i.e. $f_1(z), f_2(z)$ which are numerically stable in the neighbourhood of the corresponding singularity.

Around the singularity $z = 0$, we have

$$\begin{aligned} f_1(z) &= {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};z\right), \\ f_2(z) &= z^{1-c}{}_2F_1\left(\begin{matrix} a-c+1, b-c+1 \\ 2-c \end{matrix};z\right). \end{aligned} \tag{1.23}$$

The fundamental pair of solutions around the singularity $z = 1$ are

$$\begin{aligned} f_1(z) &= {}_2F_1\left(\begin{matrix} a, b \\ a+b-c+1 \end{matrix}; 1-z\right), \\ f_2(z) &= (1-z)^{c-a-b}\, {}_2F_1\left(\begin{matrix} c-a, c-b \\ c-a-b+1 \end{matrix}; 1-z\right), \end{aligned} \tag{1.24}$$

and around the singularity $z = \infty$, we have

$$\begin{aligned} f_1(z) &= z^{-a} {}_2F_1\left(\begin{matrix} a, a-c+1 \\ a-b+1 \end{matrix}; \frac{1}{z}\right), \\ f_2(z) &= z^{-b} {}_2F_1\left(\begin{matrix} b, b-c+1 \\ b-a+1 \end{matrix}; \frac{1}{z}\right). \end{aligned} \tag{1.25}$$

For the cases when $c, c-a-b, a-b \in \mathbb{Z}$, see ([1], [41] and §15.10 in [28]).

1.2.2 Gauss' contiguous relations

Any two hypergeometric functions are known to be *contiguous* if they satisfy the following properties;

1. they have the same power series variable,
2. any two of the parameters are pairwise equal and
3. the third parameter differ by 1.

Any two of the six hypergeometric functions of the form:

$${}_2F_1\left(\begin{matrix} a\pm 1, b \\ c \end{matrix}; z\right), \qquad {}_2F_1\left(\begin{matrix} a, b\pm 1 \\ c \end{matrix}; z\right), \qquad {}_2F_1\left(\begin{matrix} a, b \\ c\pm 1 \end{matrix}; z\right), \tag{1.26}$$

and the hypergeometric function ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)$ can be expressed as a linear combination of each other. There are 15 such relations which are known as *contiguous relations.* Applying these relations repeatedly, for any $k, l, m \in \mathbb{Z}$, the hypergeometric function

$${}_2F_1\left(\begin{matrix} a+k, b+l \\ c+m \end{matrix}; z\right), \tag{1.27}$$

can be expressed as a linear combination of ${}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; z\right)$ and any of its contiguous functions, where the coefficients are the rational functions of a, b, c and z.

Example 1.5.

$$(b-a)\,{}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; z\right) + a\,{}_2F_1\left(\begin{matrix}a+1, b\\ c\end{matrix}; z\right) - b\,{}_2F_1\left(\begin{matrix}a, b+1\\ c\end{matrix}; z\right) = 0,$$
$$c(1-z)\,{}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; z\right) - c\,{}_2F_1\left(\begin{matrix}a-1, b\\ c\end{matrix}; z\right) + (c-b)z\,{}_2F_1\left(\begin{matrix}a, b\\ c+1\end{matrix}; z\right) = 0. \tag{1.28}$$

For more of contiguous relations we refer to Section 15.5(ii) in [28].

1.2.3 Integral representations

The hypergeometric function (1.12) can be represented by the following integral representations which are valid for $|\mathrm{ph}\,(1-z)| < \pi$.

$$\mathbf{F}\left(\begin{matrix}a, b\\ c\end{matrix}; z\right) = \frac{1}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}\,(1-t)^{c-b-1}}{(1-zt)^a}\,dt, \tag{1.29}$$

where $\Re(c) > \Re(b) > 0$. This integral representation is due to Euler (1769).

$$\mathbf{F}\left(\begin{matrix}a, b\\ c\end{matrix}; z\right) = \frac{\Gamma(b-c+1)}{2\pi i\Gamma(b)} \int_0^{(1+)} \frac{t^{b-1}\,(t-1)^{c-b-1}}{(1-zt)^a}\,dt, \tag{1.30}$$

where $c-b \neq 1, 2, 3, \cdots, \Re(b) > 0$. The contour in the complex plane starts from $t = 0$, makes a loop around $t = 1$ and goes back to its original position. The branch point $t = \frac{1}{z}$ lies outside the contour of integration.

$$\mathbf{F}\left(\begin{matrix}a, b\\ c\end{matrix}; z\right) = e^{-b\pi i}\frac{\Gamma(-b+1)}{2\pi i\Gamma(c-b)} \int_\infty^{(0+)} \frac{t^{b-1}\,(t+1)^{a-c}}{(t-zt+1)^a}\,dt, \tag{1.31}$$

where $b \neq 1, 2, 3, \cdots, \Re(c-b) > 0$. Here the path of integration is a loop contour starting from $t = \infty$, encircling $t = 0$ and terminating at $t = \infty$, keeping in mind that the branch points $t = \frac{1}{z-1}$ and $t = -1$ lie outside the integration contour.

$$\mathbf{F}\left(\begin{matrix}a, b\\ c\end{matrix}; z\right) = e^{-b\pi i}\frac{\Gamma(-b+1)}{2\pi i\Gamma(c-b)} \int_1^{(0+)} \frac{t^{b-1}\,(1-t)^{c-b-1}}{(1-zt)^a}\,dt, \tag{1.32}$$

where $b \neq 1, 2, 3, \cdots, \Re(c-b) > 0$. In the above integral the point $t = \frac{1}{z}$ lies outside the contour of integration which is a loop around $t = 0$ starting and ending at $t = 1$.

In all the integrals discussed above, the contours of integration are taken with positive orientation and the integrands are continuous functions of t on the path of integration except possibly at the end points.

More integral representations of the hypergeometric functions are given in Section 15.6 in [28].

1.2.4 Linear transformations and Kummer's 24 solutions

The Gauss hypergeometric function (1.12) satisfies a number of linear transformations. The main linear transformations which are used to distribute the parameters to others are given below which were introduced by Pfaff (1797) and Euler (1794)

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) &= (1-z)^{-a}\, {}_2F_1\left(\begin{matrix} a, c-b \\ c \end{matrix}; \frac{z}{z-1}\right), \quad \text{(Pfaff)} \qquad (1.33) \\ &= (1-z)^{-b}\, {}_2F_1\left(\begin{matrix} c-a, b \\ c \end{matrix}; \frac{z}{z-1}\right), \qquad (1.34) \\ &= (1-z)^{c-a-b}\, {}_2F_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix}; z\right). \quad \text{(Euler)} \qquad (1.35) \end{aligned}$$

For the proof of these identities see [1].

Using the linear transformation Eqs. ((1.33)-(1.35)), we can transform the three pairs of the fundamental solutions Eqs. ((1.23)-(1.25)) of the hypergeometric differential equation (1.21) into 18 other solutions which lead to a total of 24 solutions of (1.21), known as the *Kummer's solutions*. Connection formulas are given in Section (15.10.(ii)) in [28]. For some of them we refer to Eqs. (3.14), (3.18), (3.119), and (3.129) in Chapter 3. For example by using the following connection formula one can check the behaviour of the hypergeometric function for large z,

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a}\, {}_2F_1\left(\begin{matrix} a, a-c+1 \\ a-b+1 \end{matrix}; \frac{1}{z}\right) \\ + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b}\, {}_2F_1\left(\begin{matrix} b, b-c+1 \\ b-a+1 \end{matrix}; \frac{1}{z}\right), \end{aligned} \qquad (1.36)$$

where $|\mathrm{ph}\,(-z)| < \pi$.

1.3 Integral Asymptotics

The general study of asymptotics looks at the behaviour of a function $F(\lambda)$ as $\lambda \to \infty$. Integral asymptotics mainly focus on the integrals of the form

$$F(\lambda) = \int_{\mathcal{C}} e^{-\lambda f(t)} g(t)\, dt, \tag{1.37}$$

(known as Laplace-type integrals) where $\mathcal{C}$ is a contour and $\lambda \to \infty$. The functions $f(t)$ and $g(t)$ are analytic along the contour of integration. The contour $\mathcal{C}$ is chosen in such a way that it avoids all the singularities and the branch points of the integrand.

The asymptotic behaviour of (1.37) is determined by the critical points of the integral which are the singularities of the f and g, the poles, branch points or the saddle points of the phase function f. To obtain the asymptotic expansion of (1.37), one needs to expand the functions f and g at those critical points and use transformations to change the integral into its standard form.

In the literature, we can find various methods which deal with the investigation of the behaviour of these kind of integrals as in (1.37) as $\lambda \to \infty$. In the next section, we discuss one of them which is known as *the saddle point method.*

1.3.1 The saddle point method

To investigate the behaviour of the integral (1.37), we make use of the saddle point method. The saddle point method is a well-known method in asymptotic analysis and it was introduced by Riemann in 1863 [35]. Later on, in 1909 Debye [8] employed this method for the analysis of Bessel functions for large order.

The main idea of the saddle point method, (also known as the method of steepest descent), is that by using the Cauchy's theorem, one can deform the contour $\mathcal{C}$ to different contours on which $\Re(f(t))$ has a minimum and $\Im f(t)$ is constant. This is chosen so that the oscillating contributions do not cancel.

In general these deformed contours pass through the points $t = t_s$ where $f'(t_s) = 0$. Such points are known as the saddle points of the phase function $f(t)$. Once the contour is deformed, to obtain the asymptotic approximation of $F(\lambda)$ in (1.37) as $\lambda \to \infty$, one needs to use the transformation

$$f(t) = f(t_s) + \tau, \tag{1.38}$$

where τ is real and can be monotonically increasing or decreasing. Using this transformation in the integral representation (1.37), we obtain

$$F(\lambda) = \sum_s e^{-\lambda f(t_s)} \int_{\tilde{\mathcal{C}}_s} e^{-\lambda\tau} g(t) \frac{dt}{d\tau}\, d\tau. \tag{1.39}$$

The new deformed contours are considered to be the steepest descent paths from the saddle points. Thus the original integral can be written as a sum of the new integrals along the steepest descent paths. The dominant saddle point is the one at which $\Re\{f(t_s)\}$ has the minimum value.

By using all these assumptions, when the main contribution is from these saddle points, one can apply the saddle point method as follows (Section 2.4.(iv) in [34])

$$\int_{\mathcal{C}} e^{-\lambda f(t)} g(t)\, dt \sim 2 \sum_j e^{-\lambda f(s_j)} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2}) b_{2n,j}}{\lambda^{n+1/2}}, \tag{1.40}$$

as $\lambda \to \infty$, where

$$b_{0,j} = \frac{g}{\sqrt{2f''}}, \qquad b_{2,j} = \frac{1}{\sqrt{2f''}} \left(\frac{g''}{f''} - \frac{f'''g'}{f''^2} + \frac{5f'''^2 g}{12 f''^3} - \frac{f^{iv} g}{4 f''^2} \right), \tag{1.41}$$

where f and g are evaluated at the saddle points s_j. To find the higher coefficients $b_{2n,j}$, one can use Eq. 2.3.18 in [34] by replacing $\lambda = 1$, $\mu = 2$ and s by $2n$. In (1.40) we sum over all the contributing saddle points.

The saddle point method is a well-known method to obtain asymptotic approximation of various special functions involving large parameters.

1.3.2 Application: Asymptotics of hypergeometric function

In this section, we obtain the asymptotic approximation of hypergeometric functions using the saddle point method. We consider the hypergeometric function

$$F(\lambda, z) = {}_2F_1\left({a, b-\lambda \atop c+\lambda} ; -z \right), \tag{1.42}$$

where we take $a = \frac{1}{2}, b = \frac{1}{8}, c = \frac{3}{4}$.

We assume that $z \in (0,1)$ and take λ real and positive. We are interested in obtaining the asymptotic expansion of (1.42) as $\lambda \to \infty$. Using the integral representation (1.31) for the hypergeometric function (1.42), we have

$$F(\lambda, z) = e^{(\lambda-1/8)\pi i}\frac{\Gamma(\lambda+\frac{3}{4})\Gamma(\lambda+\frac{7}{8})}{2\pi i\Gamma(2\lambda+\frac{5}{8})}\int_{\infty}^{(0+)}\frac{\tau^{-3/4-\lambda}\left(1+\tau\right)^{-1/4-\lambda}}{\left(\tau+z\tau+1\right)^{1/2}}\,d\tau. \tag{1.43}$$

Substituting $\tau = \frac{t-1}{2}$ in its integral representation (1.43), we obtain

$$F(\lambda, z) = \frac{2^{2\lambda+5/8}\Gamma(\lambda+\frac{3}{4})\Gamma(\lambda+\frac{7}{8})}{2\pi i\Gamma(2\lambda+\frac{5}{8})\left(z+1\right)^{1/2}}\int_{-i\infty}^{i\infty}\frac{\left(1-t\right)^{-7/8}\left(1+t\right)^{-1/4}}{\left(1-t^2\right)^{\lambda}\left(t-\frac{z-1}{z+1}\right)^{1/2}}\,dt. \tag{1.44}$$

Expressing the integral (1.44) in the form of (1.37), the phase function will be

$$f(t) = \ln\left(1-t^2\right), \tag{1.45}$$

and

$$g(t) = \frac{\left(1-t\right)^{-7/8}\left(1+t\right)^{-1/4}}{\left(t-\frac{z-1}{z+1}\right)^{1/2}}. \tag{1.46}$$

By equating $f'(t) = 0$, the saddle point is located at $t = 0$. The branch points of the integral are at $t = \pm 1$ and $t_c = \frac{z-1}{z+1}$. We choose branch cuts of the integral (1.44) along $(-\infty, -1]$ and $[1, \infty)$.

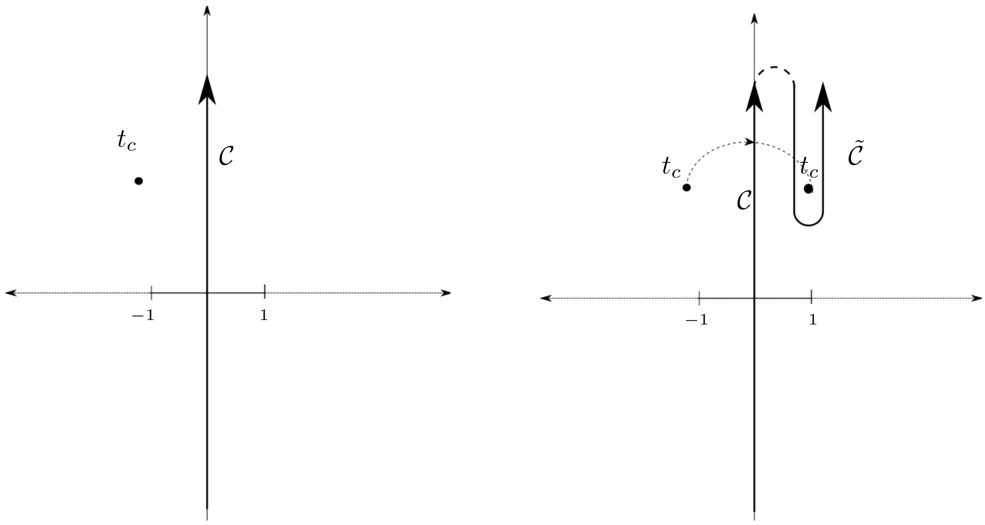

Figure 1.1: The steepest descent contour. The figure (right) depicts the situation when the branch point t_c crosses the contour $\mathcal{C}$ and switches on a new loop contour $\tilde{\mathcal{C}}$ around t_c.

Since we take $z \in (0, 1)$ and $\operatorname{ph} \lambda = 0$, the steepest descent contour will be the imaginary axis as shown in Figure 1.1(left), the branch point t_c will be bounded away from the path of integration and thus the main contribution will be from the saddle point at $t = 0$. By applying the saddle point method (1.40), we obtain

$$F(\lambda, z) \sim \frac{2^{2\lambda+5/8}\Gamma\left(\lambda+\frac{3}{4}\right)\Gamma\left(\lambda+\frac{7}{8}\right)}{\pi i\Gamma\left(2\lambda+\frac{5}{8}\right)(z+1)^{1/2}} \sum_{n=0}^{\infty} \Gamma(n+\tfrac{1}{2}) \frac{b_{2n}}{\lambda^{n+1/2}}, \tag{1.47}$$

as $\lambda \to \infty$. Using (1.41) we obtain

$$b_0 = \frac{i}{2}\left(\frac{1-z}{z+1}\right)^{-1/2}, \tag{1.48}$$

Thus we obtain

$$F(\lambda, z) \sim \frac{2^{-3/8+2\lambda}\Gamma\left(\lambda+\frac{3}{4}\right)\Gamma\left(\lambda+\frac{7}{8}\right)}{\Gamma\left(2\lambda+\frac{5}{8}\right)\sqrt{\lambda\pi}\,(1-z)^{1/2}}. \tag{1.49}$$

Now if we move z in such a way that the critical branch point $t_c = \frac{z-1}{z+1}$ crosses the steepest descent path, then it will switch on a new contribution which will be a loop contour starting from $t = \infty$, encircling around the branch point $t = t_c$ goes back to its starting position as shown on Figure 1.1(right). (This is expressed in the integral representation

(1.51)). This contribution will be switched on when the Stokes phenomenon takes place.

The role of Stokes phenomenon has been of great importance to study the behaviour of the asymptotic expansions for the last 20 years. The Stokes phenomenon, named after Sir George Gabriel Stokes [38], refers to the fact that the asymptotic expansion of a function can only be uniform in $\operatorname{ph} z$ if we can construct an exponentially small correction as certain directions are traversed in the complex plane. These directions are known as *Stokes lines*. Stokes lines are the lines when the imaginary part of the two contributing terms are equal. At these lines one term of behaviour, say $w_1 \approx e^{\lambda f_1(z)}$, is maximally dominant over $w_2 \approx e^{\lambda f_2(z)}$, since, by definition, $\lambda(f_1(z) - f_2(z))$ is positive along the Stokes curve. The Stokes lines are separated by other directions known as *anti-Stokes lines* at which the complementary functions switches from being dominant to subdominant.

We will show below that in our example, the Stokes curve is located at

$$\Im\left(\lambda \ln\left(\frac{4z}{(z+1)^2}\right)\right) = 0. \tag{1.50}$$

In the case that $\operatorname{ph}\lambda = 0$, this is the unit circle.

The new term which is switched on due to the Stokes phenomenon is defined by the following integral representation:

$$T = \frac{2^{2\lambda+5/8}\,(z+1)^{-1/2}\,\Gamma(\lambda+\frac{3}{4})\Gamma(\lambda+\frac{7}{8})}{2\pi i\Gamma(2\lambda+\frac{5}{8})}\int_{i\infty}^{(t_c+)} \frac{(1-t)^{-7/8}\,(1+t)^{-1/4}}{(1-t^2)^{\lambda}\left(t-\frac{z-1}{z+1}\right)^{1/2}}\,dt, \tag{1.51}$$

which is equivalent to

$$T = \frac{\Gamma(\lambda+\frac{3}{4})\Gamma(\lambda+\frac{7}{8})\,(z+1)^{2\lambda+1/8}}{\Gamma(2\lambda+\frac{9}{8})\sqrt{\pi}z^{\lambda+1/4}}\,{}_2F_1\left(\begin{matrix}1/2, -\lambda+1/4\\ 2\lambda+9/8\end{matrix}; \frac{z+1}{z}\right). \tag{1.52}$$

Using the integral representation (Eq. 15.6.1 in [28]) for the hypergeometric function in (1.52), we obtain

$$T = \frac{\Gamma(\lambda+\frac{3}{4})\,(z+1)^{2\lambda-3/8}}{\Gamma(\lambda+\frac{1}{4})\sqrt{\pi}z^{\lambda-1/4}}\int_0^1 e^{-\lambda f(t)}g(t)\,dt, \tag{1.53}$$

where

$$f(t) = -\ln(t(1-t)), \tag{1.54}$$

$$g(t)=\frac{t^{-3/4}\,(1-t)^{-1/8}}{\left(\frac{z}{z+1}-t\right)^{1/2}}. \tag{1.55}$$

Note that the saddle point is located at $t=1/2$. The branch points of the integral are located at $t=0,1,\frac{z}{z+1}$. When z is close to -1, then the main contribution will come from the saddle point and thus again by applying the saddle point method, we obtain

$$T\sim\frac{2^{1-2\lambda}\Gamma(\lambda+\frac{3}{4})\,(z+1)^{2\lambda-3/8}}{\Gamma(\lambda+\frac{1}{4})\sqrt{\pi}z^{\lambda-1/4}}\sum_{n=0}^{\infty}\Gamma(n+1/2)\frac{b_{2n}}{\lambda^{n+1/2}}, \tag{1.56}$$

as $\lambda\to\infty$, given that

$$b_0=2^{-5/8}\left(\frac{z-1}{z+1}\right)^{-1/2}. \tag{1.57}$$

Substituting the value of b_0 in (1.56), the first term of the asymptotic approximation is given as

$$T\sim\frac{2^{3/8}z^{1/4}\,(z+1)^{1/8}\,\Gamma(\lambda+\frac{3}{4})}{\Gamma(\lambda+\frac{1}{4})\lambda^{1/2}\,(z-1)^{1/2}}\left(\frac{(z+1)^2}{4z}\right)^{\lambda}. \tag{1.58}$$

The Stokes curve for $\mathrm{ph}\,\lambda=0$ is shown in Figure 1.2

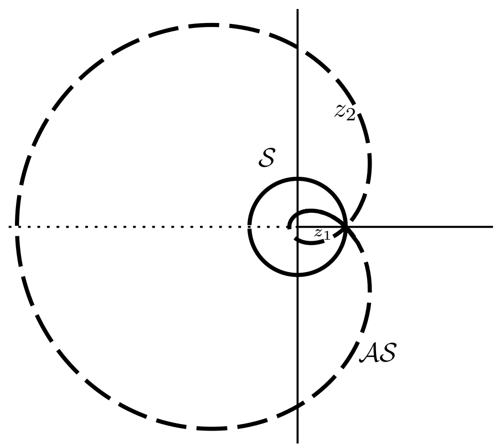

Figure 1.2: Stokes (solid line $\mathcal{S}$) and anti-Stokes lines (dashed line $\mathcal{AS}$) for $\mathrm{ph}\,\lambda=0$. Inside the bold line (Stokes curve), the asymptotic expansion (1.49) is dominant.

The dashed lines are the anti-Stokes curves where the two functions are of the same order. Inside the bold circle the main contribution will come from the original saddle point. When the Stokes curve is crossed then the new contribution T is born, and the asymptotic approximation will then be the sum of both the asymptotic expansions. When the anti-Stokes line is crossed then the term T will become dominant.

Thus for $|z| > 1$, we obtain

$$F(\lambda, z) \sim \frac{2^{2\lambda-3/8}\Gamma\left(\lambda+\frac{3}{4}\right)\Gamma\left(\lambda+\frac{7}{8}\right)}{\Gamma\left(2\lambda+\frac{5}{8}\right)\sqrt{\pi\lambda}\,(1-z)^{1/2}} + \frac{2^{3/8}z^{1/4}\,(z+1)^{1/8}\,\Gamma(\lambda+\frac{3}{4})}{\Gamma(\lambda+\frac{1}{4})\lambda^{1/2}\,(z-1)^{1/2}}\left(\frac{(z+1)^2}{4z}\right)^{\lambda}, \tag{1.59}$$

as $\lambda \to \infty$.

The asymptotic approximation of the hypergeometric function (1.42) as $\lambda \to \infty$ and $|\mathrm{ph}\,\lambda| \leqslant \pi/2$ will be discussed in detail in §3.4 in Chapter 3.

Example 1.6 (Numerical illustration)**.** If we take z inside the Stokes line $\mathcal{S}$, such that $z \in (0,1)$ i.e. if $z_1 = 0.6$ and $\lambda = 100$, then the exact value of our hypergeometric function (1.42) is 1.549116227. Only taking the first term of the asymptotic approximation (1.59), we obtain 1.581696918. At this point, since the Stokes curve is not crossed so the second term in (1.59) is not switched on.

Now when z crosses the Stokes lines $\mathcal{S}$ and is close to the anti-Stokes line $\mathcal{AS}$ i.e. if $z_2 = 1.0 + 2.7i$ (as shown in Figure 1.2), then the asymptotic approximation of (1.42) will include the contribution of the extra term T. The exact value of the function is $0.3839418560 + 0.5883753167i$ and the value of extra term is $T = -0.04377999524 + 0.1572916637i$. The asymptotic approximation of the first term at the saddle point $t = 0$ is $0.4304833749 + 0.4304833749i$ and hence we need to add the asymptotic approximation of the second term i.e. $-0.04409083416 + 0.1574305296i$. The combination of these two approximations gives us $0.3863925407 + 0.5879139045i$, where the error is 0.003549486921.

1.3.3 Uniform asymptotic expansions

In the situation when the critical points depend on an auxiliary parameter, the behaviour of the asymptotic approximations can change dramatically. One needs to adopt different methods to obtain the asymptotic approximations which are uniform with respect to the auxiliary parameter. Uniform asymptotic expansions are useful in describing the transition of behaviour of a function. For example, for the situations when a coalescence happens between two saddle points or if a branch point approaches a saddle point due to the existence of the additional parameters, then we apply different methods to tackle these kind of problems. In general higher transcendental functions such as Airy function, Bessel functions, parabolic cylinder functions *etc.*, are used in uniform asymptotic expansions.

Definition 1.4 (Uniform asymptotic approximation)**.** In *uniform asymptotics*, we deal with an auxiliary parameter known as uniformity parameter ζ which appears in the asymptotic sequence $\{\phi_n(z,\zeta)\}, n = 0, 1, 2, \cdots,$

$$\frac{\phi_{n+1}(z,\zeta)}{\phi_n(z,\zeta)} \to 0, \qquad \text{as} \quad z \to \infty, \tag{1.60}$$

uniformly *w.r.t* $\zeta \in \mathcal{D}$. Then we say that $F(z,\zeta)$ has a uniform asymptotic expansion $\sum_{n=0}^{\infty} a_n(\zeta)\phi_n(z,\zeta)$ as $z \to \infty$, $\zeta \in \mathcal{D}$, *i.e.*

$$F(z,\zeta) = \sum_{n=0}^{N-1} a_n(\zeta)\phi_n(z,\zeta) + R_N(z,\zeta), \tag{1.61}$$

where

$$R_N(z,\zeta) = \mathcal{O}(\phi_N(z,\zeta)), \tag{1.62}$$

for large z and $\zeta \in \mathcal{D}$.

For more detail see Chapter VII in [47].

1.3.4 Bleistein's Method: Two coalescing saddle points

We consider the integral of the form

$$F(\lambda,\zeta) = \int_{\mathcal{C}} e^{\lambda f(t,\zeta)} g(t,\zeta)\, dt. \tag{1.63}$$

Here the location of the critical points depend on the additional parameter ζ. The asymptotic approximation of the integral (1.63) as $\lambda \to \infty$, which is uniform with respect to ζ, depends on the relevant critical points.

Now we suppose that the integral (1.63) has two saddle points located at $t = t_\pm$ which depend on ζ. In this case both the saddle points will contribute and by applying the saddle point method, we obtain

$$F(\lambda,\zeta) \sim \pm g(t_+)e^{\lambda f(t_+)}\left(-\frac{2\pi}{\lambda f''(t_+)}\right)^{1/2} \pm g(t_-)e^{\lambda f(t_-)}\left(-\frac{2\pi}{\lambda f''(t_-)}\right)^{1/2}, \tag{1.64}$$

as $\lambda \to \infty$. The $\pm$ sign in (1.64) depends on the direction of the contour.

If there exists a critical value ζ_0 such that when $\zeta = \zeta_0$, the two saddle points coalesce at $t = t_0$, then we will have saddle points of multiplicity 2, *i.e.*,

$$\begin{aligned} f'(t_0,\zeta_0) &= f''(t_0,\zeta_0) = 0, \\ f'''(t_0,\zeta_0) &\neq 0. \end{aligned} \tag{1.65}$$

One can observe that when $\zeta = \zeta_0$, then the asymptotic approximation (1.64) breaks down.

To obtain the asymptotic approximation of (1.63), we expand the functions f and g around the saddle point $t = t_0$ and obtain

$$F(\lambda,\zeta) \sim \int_{\mathcal{C}} e^{\lambda\left(f(t_0)+\frac{1}{3!}f'''(t_0)(t-t_0)^3\right)} g(t_0)\, dt. \tag{1.66}$$

By change of variables *i.e.*, using $\tau = 3^{1/3}\left(\frac{1}{3!}f'''(t_0)\right)^{1/3}(t-t_0)$, the integral (1.66) is reduced to

$$F(\lambda,\zeta) \sim \frac{1}{3^{1/3}}\left(\frac{1}{3!}f'''(t_0)\right)^{-1/3} e^{\lambda f(t_0)} g(t_0) \int_{\tilde{\mathcal{C}}} e^{\lambda\frac{\tau^3}{3}}\, d\tau. \tag{1.67}$$

Now by using the integral representation of Airy function (see [32])

$$\mathrm{Ai}\,(x) = \frac{1}{2\pi i}\int_{\infty e^{-\pi i/3}}^{\infty e^{+\pi i/3}} e^{\frac{1}{3}t^3 - xt}\, dt, \tag{1.68}$$

where $x \in \mathbb{C}$, and using the fact that

$$\mathrm{Ai}\,(0) = \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)}, \tag{1.69}$$

we obtain the following asymptotic approximation

$$F(\lambda,\zeta) \sim K g(t_0) e^{\lambda f(t_0)} \Gamma\left(\frac{4}{3}\right)\left(\frac{3!}{\lambda f'''(t_0)}\right)^{1/3}, \tag{1.70}$$

as $\lambda \to \infty$. The constant K depends on the contour $\tilde{\mathcal{C}}$. One can observe that we obtain two very different asymptotic approximations (1.64) and (1.70). When $\zeta \to \zeta_0$, the asymptotic approximation (1.64) will have a singularity. Moreover, it is noticeable that the order of λ changes discontinuously from $\frac{1}{2}$ in (1.64) to $\frac{1}{3}$ in the asymptotic approximation (1.70). Thus the asymptotic approximation which we obtain via the saddle point method is not uniform as $\zeta \to \zeta_0$. (For more detail we refer to Chapter VII in [47]).

Hence, to obtain asymptotic approximations which holds uniformly when the two saddle points coalesce or when the saddle point coalesce with the branch point or the end point, we adopt the *Bleistein method*. We will now illustrate some of the main steps in the process of obtaining uniform asymptotic expansions via the Bleistein method. Note that in the case of an integral in which two coalescing saddles dominate the asymptotics, one must convert the integral to its canonical form.

In order to express the integral (1.63) in its canonical form, we consider the cubic transformation [6] suggested by Chester, Friedman and Ursell in 1957 that exhibits two coalescing saddle points. The transformation is

$$f(t,\zeta) = \tfrac{1}{3}u^3 - \zeta u + \eta, \tag{1.71}$$

where the coefficients ζ and η can be determined using the fact that the transformation is analytic at the saddle points $t = t_+$ and $t = t_-$. The new phase function will have the saddle points at $u = \pm\sqrt{\zeta}$ in the complex u-plane. From the transformation (1.71), the coefficients are

$$\zeta^{3/2} = \tfrac{3}{4}\left(f(t_-) - f(t_+)\right), \tag{1.72}$$

and

$$\eta = \tfrac{1}{2}\left(f(t_-) + f(t_+)\right). \tag{1.73}$$

Substituting the cubic transformation (1.71) in the integral (1.63), we obtain

$$F(\lambda,\zeta) = e^{\eta\lambda} \int_{\tilde{\mathcal{C}}} e^{\lambda\left(\frac{1}{3}u^3 - \zeta u\right)} G_0(u)\, du, \tag{1.74}$$

where $\tilde{\mathcal{C}}$ is the image of the contour $\mathcal{C}$ and

$$G_0(u) = g(t)\frac{dt}{du}, \tag{1.75}$$

where we assume that $\lambda > 0$. Note that the phase function has saddle points at $t = \pm\sqrt{\zeta}$. In the case that $\zeta > 0$ there exists a steepest descent path just passing through one saddle point, and in the case that $\zeta < 0$ the steepest descent path will pass through both saddle points. See Figure 1.3.

It follows that in the case we want an asymptotic approximation for $\lambda \to \infty$, that holds for ζ close to origin then both the saddle points should contribute.

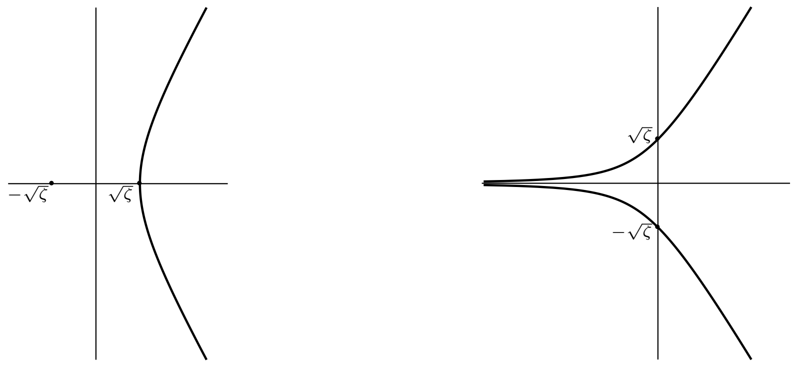

Figure 1.3: The left figure is for $\zeta > 0$. In that case the steepest descent path of integration in (1.74) passes through only one saddle point. The figure on the right is for the case $\zeta < 0$.

The well-known method to obtain the uniform asymptotic approximation is known as *Bleistein's Method* [5], which was introduced by Bleistein in 1966. It is a multipoint expansion of $G_n(u)$ about the relevant critical points and is based on the following recursive scheme with a special integration by parts

$$\begin{aligned} G_n(u) &= p_n + q_n u + (u^2 - \zeta)H_n(u), \\ G_{n+1}(u) &= -\frac{d}{du}H_n(u), \end{aligned} \tag{1.76}$$

for $n = 0, 1, 2, \cdots$. The coefficients p_n and q_n using (1.76) at the two saddle points

$u = \pm\sqrt{\zeta}$ are:

$$p_n = \frac{G_n(\sqrt{\zeta}) + G_n(-\sqrt{\zeta})}{2}, \qquad q_n = \frac{G_n(\sqrt{\zeta}) - G_n(-\sqrt{\zeta})}{2\sqrt{\zeta}}. \tag{1.77}$$

It is important that in this process, we do not generate any new singularities. It is constructed in such a way that the growth of $G_n(u)$ at $u = \infty$ is the same as $G_0(u)$. If we use (1.76) in (1.74) and by integrating it N-times by parts we obtain the following uniform asymptotic approximation in terms of Airy functions

$$F(\lambda,\zeta) = e^{\eta\lambda}\left(\mathrm{Ai}\,(\lambda^{2/3}\zeta)\sum_{s=0}^{N-1} p_s\lambda^{-s-1/3} - \mathrm{Ai}\,'(\lambda^{2/3}\zeta)\sum_{s=0}^{N-1} q_s\lambda^{-s-2/3} + R_N(\lambda,\zeta)\right), \tag{1.78}$$

where

$$R_N(\lambda,\zeta) = \frac{\lambda^{-N}}{2\pi i}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} e^{\lambda\left(\frac{1}{3}u^3-\zeta u\right)} G_N(u)\,du. \tag{1.79}$$

In the uniform asymptotic approximation (1.78), $\mathrm{Ai}\,(x)$ is the *Airy function* and $\mathrm{Ai}\,'(x)$ is its derivate. The integral representation for the Airy function is given in (1.68).

To show that the expansion in (1.78) has an asymptotic property one has to show that there exists a positive constant K_N such that

$$\left|\lambda^N R_N(\lambda,\zeta)\right| \leqslant K_N\left(\lambda^{-1/3}\left|\mathrm{Ai}\,\left(\zeta\lambda^{2/3}\right)\right| + \lambda^{-2/3}\left|\mathrm{Ai}\,'\left(\zeta\lambda^{2/3}\right)\right|\right). \tag{1.80}$$

Usually (see [47], page 371) one splits the proof in two parts: for the case $\left|\zeta\lambda^{2/3}\right| \geqslant \rho$ one can check the contributions of the saddle points, and for the case $\left|\zeta\lambda^{2/3}\right| \leqslant \rho$ one makes the following observation. Note that in this case the Airy function in (1.78) and its derivative are just bounded functions. It is easy to show that in this case $\lambda^N R_N(\lambda,\zeta) = \mathcal{O}\left(\lambda^{-1/3}\right)$, as $\lambda \to \infty$, but this is not sufficient since the Airy function in (1.80) could be zero. One extra integration by parts is needed:

$$\lambda^N R_N(\lambda,\zeta) = p_N\lambda^{-1/3}\mathrm{Ai}\left(\zeta\lambda^{2/3}\right) - q_N\lambda^{-2/3}\mathrm{Ai}'\left(\zeta\lambda^{2/3}\right) + \lambda^N R_{N+1}(\lambda,\zeta), \tag{1.81}$$

and since $\lambda^N R_{N+1}(\lambda,\zeta) = \mathcal{O}\left(\lambda^{-4/3}\right)$, as $\lambda \to \infty$, bound (1.78) holds. Here we also use the fact that the Airy function and its derivative cannot be zero at the same time, since they are the solution of second order linear differential equations.

1.3.5 Application: Uniform Asymptotic Approximation of Hypergeometric function

Now we consider the hypergeometric function of the form

$$F(\lambda, z) = {}_2F_1\left(\begin{matrix} a+\lambda, b-\lambda \\ c \end{matrix}; -z\right), \tag{1.82}$$

where we choose $a = b = \frac{1}{3}$, $c = \frac{3}{5}$. Using the integral representation defined in (3.38) in Chapter 3, we have

$$F(\lambda, z) = \frac{\Gamma(\frac{3}{5})\Gamma(\lambda + \frac{11}{15})}{2\pi i \Gamma(\lambda + \frac{1}{3})} \int_{-\infty}^{(0+)} \frac{t^{-\lambda - \frac{11}{15}} (1-t)^{-\lambda - \frac{4}{15}}}{(z+1-t)^{-\lambda + \frac{1}{3}}} \, dt. \tag{1.83}$$

The path of integration starts from $t = e^{-\pi i}\infty$ and after making a loop around $t = 0$, it goes back to $t = e^{\pi i}\infty$. The points $t = 1$ and $t = z + 1$ lie outside the contour. Writing the integral in its standard form, we have

$$F(\lambda, z) = \frac{L}{2\pi i} \int_{-\infty}^{(0+)} e^{\lambda f(t)} g(t) \, dt, \tag{1.84}$$

where the phase function is

$$f(t) = \ln\left(\frac{z+1-t}{1-t}\right) - \ln t, \tag{1.85}$$

with

$$g(t) = \frac{t^{-11/15} (1-t)^{-4/15}}{(z+1-t)^{1/3}}, \tag{1.86}$$

and

$$L = \frac{\Gamma(\frac{3}{5})\Gamma(\lambda + \frac{11}{15})}{\Gamma(\lambda + \frac{1}{3})}. \tag{1.87}$$

We choose the branch cuts between $t = z + 1$ and $t = 1$ and along $(-\infty, 0]$. The saddle points are located at

$$sp_{\pm} = z + 1 \pm \sqrt{z(z+1)}. \tag{1.88}$$

We note that when $z \to 0$, then the two saddle points $sp_{\pm}$ and two of the branch points coalesce at $t = 1$. Although, four points coalesce with each other, one should see this as the coalescence of $sp_{\pm}$ with $t = z + 1$, since $t = z + 1$ is one of the end points of the steepest descent paths and $f(z+1) = -\infty$ and $f(1) = +\infty$. Hence, the value of f at these branch points is considerably different.

To obtain the uniform asymptotic approximation, we use the transformation [40]

$$f(t) = \tau + \frac{\zeta^2}{4\tau} + \eta. \tag{1.89}$$

The saddle points $sp_\pm$ in the t-plane should correspond to the saddle points in the τ-plane i.e. $\tau = \mp\zeta/2$. Also we have

$$f(sp_\pm) = \mp\zeta + \eta. \tag{1.90}$$

Thus, we obtain

$$\zeta = \ln(1 + 2z + 2\sqrt{z(z+1)}), \tag{1.91}$$

and $\eta = 0$.

Now if $z = 0$, then $\zeta = 0$ and thus we have $f(t) = -\ln t = \tau$. It follows that as $\tau \to 0$, we have $t \sim 1 - \tau$ and $g(t) = t^{-11/15}(1-t)^{-3/5} \sim \tau^{-3/5}$.
Thus using the transformation (1.89) in the integral (1.84), we have

$$F(\lambda, z) = \frac{L}{2\pi i} \int_{-\infty}^{(0+)} e^{\lambda\left(\tau + \frac{\zeta^2}{4\tau}\right)} G_0(\tau)\tau^{-3/5}\, d\tau, \tag{1.92}$$

where

$$G_0(\tau) = g(t)\frac{dt}{d\tau}\tau^{3/5}. \tag{1.93}$$

The values of $G_0(\tau)$ at the two saddle points $\tau = \mp\zeta/2$ are:

$$G_0(\mp\zeta/2) = 2^{-1/10}(z+1)^{-17/60} z^{-1/20}\zeta^{1/10}. \tag{1.94}$$

To obtain the asymptotic approximation, we use *Bleistein's method* [5] *i.e.*, we substitute

$$G_n(\tau) = p_n + \frac{q_n}{\tau} + \left(1 - \frac{\zeta^2}{4\tau^2}\right) H_n(\tau), \tag{1.95}$$

and

$$G_{n+1}(\tau) = -\tau^{3/5}\frac{d}{d\tau}\left(\tau^{-3/5}H_n(\tau)\right). \tag{1.96}$$

The two coefficients p_n and q_n can be found using (1.95) at the two saddle points:

$$p_n = \frac{G_n\left(\frac{\zeta}{2}\right) + G_n\left(-\frac{\zeta}{2}\right)}{2}, \qquad q_n = \zeta\frac{G_n\left(\frac{\zeta}{2}\right) - G_n\left(-\frac{\zeta}{2}\right)}{4}. \tag{1.97}$$

Now for $n = 0$, we substitute (1.95) in (1.92) and obtain

$$F(\lambda, z) = \frac{L}{2\pi i}\int_{-\infty}^{(0+)} e^{\lambda\left(\tau+\frac{\zeta^2}{4\tau}\right)}\left(p_0 + \frac{q_0}{\tau} + \left(1 - \frac{\zeta^2}{4\tau^2}\right)H_0(\tau)\right)\tau^{-3/5}\,d\tau. \tag{1.98}$$

Using the integral representation for the Bessel function [33]:

$$\left(\frac{\zeta}{2}\right)^{-\nu} I_\nu(\zeta\lambda) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{\lambda\left(\tau+\frac{\zeta^2}{4\tau}\right)}\frac{d\tau}{\tau^{\nu+1}}, \tag{1.99}$$

and integrating by parts (1.98), we obtain

$$\begin{aligned} F(\lambda, z) = {} & L\left(p_0\left(\frac{\zeta}{2}\right)^{2/5} I_{-2/5}(\zeta\lambda) + q_0\left(\frac{\zeta}{2}\right)^{-3/5} I_{3/5}(\zeta\lambda)\right) \\ & + \frac{L}{2\pi i}\int_{-\infty}^{(0+)} e^{\lambda\left(\tau+\frac{\zeta^2}{4\tau}\right)} G_1(\tau)\tau^{-3/5}\,d\tau. \end{aligned} \tag{1.100}$$

Continuing this process N-times we obtain

$$\begin{aligned} F(\lambda, z) = {} & L\left(\left(\frac{\zeta}{2}\right)^{2/5} I_{-2/5}(\zeta\lambda)\sum_{j=0}^{n-1}\frac{p_j}{\lambda^j} + \left(\frac{\zeta}{2}\right)^{-3/5} I_{3/5}(\zeta\lambda)\sum_{j=0}^{n-1}\frac{q_j}{\lambda^j}\right) \\ & + \lambda^{-N}\frac{L}{2\pi i}\int_{-\infty}^{(0+)} e^{\lambda\left(\tau+\frac{\zeta^2}{4\tau}\right)} G_N(\tau)\tau^{-3/5}\,d\tau. \end{aligned} \tag{1.101}$$

Example 1.7 (Numerical illustration)**.** If we take z close to 0 i.e. $z = 0.003$ and $\lambda = 100$, then the exact value of our hypergeometric function (1.82) is 20234.61105. The corresponding value of ζ is 0.1094898129. The values of the first coefficients of Bleistein's method are $p_0 = 0.9991017313$ and $q_0 = 0$. Thus the numerical value of the uniform asymptotic expansion is 20236.58147 where the error obtained is given by 0.0000973692.

1.4 Asymptotics of Hypergeometric function

The study of asymptotic approximations of the Gauss hypergeometric functions with large parameters has a long history. This kind of problem was first investigated by Laplace where he derived for the Legendre polynomials the simple approximation

$$P_n(\cos\theta) \sim \left(\frac{2}{n\pi\sin\theta}\right)^{1/2} \cos\left((n+\tfrac{1}{2})\theta - \tfrac{1}{4}\pi\right), \tag{1.102}$$

as $n \to \infty$. and $0 < \theta < \pi$ (see [46]). Later Darboux modified the work of Laplace and also investigated the asymptotic properties of Jacobi polynomials which can be expressed in terms of the hypergeometric function ${}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \frac{1-x}{2}\right)$, where n is a large positive integer.

Riemann actually introduced the steepest descent method to obtain approximations for these hypergeometric functions with a large parameter. See [36], pp. 424–430.

In 1918, Watson [46] also discussed the asymptotic behaviour of the Gauss hypergeometric functions with large parameters using the method of steepest descent. In particular he studied the cases

$${}_2F_1\left(\begin{matrix}a+\lambda, b-\lambda\\ c\end{matrix}; \frac{1-z}{2}\right), \qquad {}_2F_1\left(\begin{matrix}a+\lambda, b+\lambda\\ c+2\lambda\end{matrix}; \frac{2}{1-z}\right), \qquad {}_2F_1\left(\begin{matrix}a, b\\ c+\lambda\end{matrix}; z\right). \tag{1.103}$$

As a special case he discussed the asymptotic expansions of generalised Legendre functions $P_n^m(z)$ and $Q_n^m(z)$ when either $|n|$ and $|m|$ are large. The results given by Watson were valid in relatively small regions of z since he only used Poincaré-type asymptotic expansions which were not valid uniformly in the neighbourhood of the critical values of z. So uniform asymptotic expansions were needed for these kinds of problems which are valid for large z regions.

Luke (1969), summarised the results by Watson and gave results for higher ${}_pF_q$ functions. In particular, he obtained asymptotic approximations of extended Jacobi polynomials which are

$${}_{p+2}F_q\left(\begin{matrix}-n, n+\lambda, a_1, \cdots, a_p\\ b_1, \cdots, b_q\end{matrix}; z\right). \tag{1.104}$$

This hypergeometric function reduces to the Jacobi polynomial if n is an integer and $p = 0$ and $q = 1$.

Later on in 2001, to obtain the solution of scattering problems arising from dielectric obstacles (see [18]), Jones studied the asymptotic properties of the hypergeometric functions [19]. He obtained a uniform asymptotic expansion of

$$ {}_2F_1\left(\begin{matrix} a+\lambda, b-\lambda \\ c \end{matrix}; \frac{1-z}{2}\right), \tag{1.105} $$

in terms of Bessel functions with error bounds. His work was based on the ideas of Olver [31] using differential equations. He also discussed the application for the Legendre functions.

Adri Olde Daalhuis obtained in 2001 and 2002 ([25] and [26] respectively) uniform asymptotic approximations of

$$ {}_2F_1\left(\begin{matrix} a, b-\lambda \\ c+\lambda \end{matrix}; -z\right), \qquad {}_2F_1\left(\begin{matrix} a+\lambda, b+2\lambda \\ c \end{matrix}; -z\right), \tag{1.106} $$

in terms of parabolic cylinder functions and Airy functions respectively as $\lambda \to \infty$. In a third paper [29] this author seemed to finalise the work on large parameter asymptotics for these hypergeometric functions. However, it became clear that the results in that paper did not cover approximations near all the critical values of z.

In 2002, Temme also discussed different cases of hypergeometric functions in [42]. The author indicated the cases which are of interest for the orthogonal polynomials, such as Jacobi, Meixner, *etc.*, and discussed the cases which still need to be done. Recently, in [**?**] and [**?**], Paris discussed the asymptotics of hypergeometric functions of the form (1.107) in which the e_j are not restricted to 0, ± 1. Only non-uniform approximations are given.

In Chapter 3 of this thesis, we will study the asymptotics of the hypergeometric functions of the form

$$ {}_2F_1\left(\begin{matrix} a+e_1\lambda, b+e_2\lambda \\ c+e_3\lambda \end{matrix}; z\right), \qquad e_j = 0, \pm 1, \qquad \lambda \to \infty. \tag{1.107} $$

There are in total 27 cases out of which the first case is the trivial one *i.e.,* $e_1 = e_2 = e_3 = 0$. See Table 1.1.

e_1	e_2	e_3	e_1	e_2	e_3	e_1	e_2	e_3
0	0	0	+1	0	0	-1	0	0
0	0	+1	+1	0	+1	-1	0	+1
0	0	-1	+1	0	-1	-1	0	-1
0	+1	0	+1	+1	0	-1	+1	0
0	+1	+1	+1	+1	+1	-1	+1	+1
0	+1	-1	+1	+1	-1	-1	+1	-1
0	-1	0	+1	-1	0	-1	-1	0
0	-1	+1	+1	-1	+1	-1	-1	+1
0	-1	-1	+1	-1	-1	-1	-1	-1

Table 1.1: All 27 cases of ${}_2F_1\left(\begin{matrix} a+e_1\lambda, b+e_2\lambda \\ c+e_3\lambda \end{matrix}; \omega\right)$ for $e_j = 0, \pm 1$

Due to the symmetry of the parameters a and b in the hypergeometric function, i.e.

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = {}_2F_1\left(\begin{matrix} b, a \\ c \end{matrix}; z\right), \tag{1.108} $$

and neglecting the trivial case, the 27 cases can be reduced into 17 cases as given in Table 1.2.

Case	e_1	e_2	e_3	Type
1	0	0	+1	
2	0	0	-1	
3	0	+1	0	
4	0	-1	0	A
5	0	+1	+1	
6	0	-1	-1	
7	+1	+1	+1	
8	-1	-1	-1	
9	+1	-1	0	
10	+1	+1	0	B
11	- 1	-1	0	
12	0	+1	-1	
13	0	-1	+1	C
14	+1	-1	+1	
15	+1	-1	-1	
16	+1	+1	-1	
17	-1	-1	+1	D

Table 1.2: 17 reduced cases of hypergeometric function

Using the Pfaff and Euler transformations (Eqs.(1.33)-(1.35)) and the linear transformations given in §15.10 in [28], the 17 cases are reduced into 4 cases as mentioned in Table 1.3 and in the last section in [46].

Case	e_1	e_2	e_3
1 (A)	0	0	$+1$
2 (B)	$+1$	-1	0
3 (C)	0	-1	$+1$
4 (D)	$+1$	$+2$	0

Table 1.3: 4 cases of hypergeometric function

Note that in Case 4 of Table 1.3 one has to take $e_2 = 2$. Cases 16 and 17 in Table 1.2 can be expressed in terms of a linear combination of two terms that are of Case 4 of Table 1.3. However in [29] it is shown that this linear combination is not always numerically stable, and that the Cases 16 and 17 should be discussed independently. Hence, in Chapter 3 of this thesis we will discuss 5 main cases. See Table 1.4. In that table the trivial subcases follow from the cases on the left via Eqs.(1.33)-(1.35) and the main sub cases are obtained by using the connection formulas.

Cases	Main subcases	Trivial subcases
1. $(0,0,1)$		$(1,1,1),(0,1,1)$
	$(0,0,-1)$	$(0,-1,-1),(-1,-1,-1)$
	$(1,0,0)$	$(-1,0,0)$
2. $(1,-1,0)$		$(1,1,0)$
		$(-1,-1,0)$
3. $(0,-1,1)$		$(1,-1,1)$
	$(0,1,-1)$	$(-1,1,-1)$
4. $(-1,-1,1)$		
5. $(1,1,-1)$		

Table 1.4: The 5 main cases and 3 main sub cases

1.5 Layout and Contibutions of the thesis

This thesis is divided into 4 chapters. The first chapter sets the sense of covering definitions and literature review.

Chapter 2 is based on obtaining new uniform asymptotic approximations for integrals with an exponentially small remainder. We illustrate how these results can be used to obtain remainder estimates in the Bleistein method. The method is created to deal with new types of integrals in which the usual methods for remainder estimates fail. As an application we obtain an asymptotic expansion for ${}_2F_1\left(\begin{matrix} a,b \\ \lambda+b \end{matrix}; -z\right)$ as $\lambda \to \infty$ in

$|\text{ph}\,\lambda| \leqslant \pi/2$ uniformly for large $|z|$. This work has been published in [21].

In Chapter 3, we discuss the uniform asymptotic approximations for the Gauss hypergeometric function

$$ {}_2F_1\left(\begin{matrix} a+e_1\lambda, b+e_2\lambda \\ c+e_3\lambda \end{matrix}; z\right), \tag{1.109}$$

$e_j = 0, \pm 1, j = 1, 2, 3$ as $|\lambda| \to \infty$, which are valid in large regions of the complex z-plane, where a, b and c are fixed. We complete the results of the three previous publications ([25], [26] and [29]), discuss all the cases and, what is new, we consider now all critical values of z. By expressing these hypergeometric functions in their integral representation, we apply Bleistein's method to obtain the uniform asymptotic approximation. For the cases where the main contribution of the integral comes from the saddle point we use the saddle point method to obtain asymptotic approximations..

Finally in the last chapter, we deal with the uniform asymptotic approximations for the monic Meixner-Sobolev polynomials $S_n(x)$. These approximations for $n \to \infty$, are uniformly valid for x/n restricted to certain intervals, and are in terms of Airy functions. We also give asymptotic approximations for the location of the zeros of $S_n(x)$, especially the small and the large zeros are discussed. As a limit case we also give a new asymptotic approximation for the large zeros of the classical Meixner polynomials.

The method is based on an integral representation in which a hypergeometric function appears in the integrand. After a transformation the hypergeometric functions can be uniformly approximated by unity, and all that remains are simple integrals for which standard asymptotic methods are used. As far as we are aware, this is the first time that standard uniform asymptotic methods are used for the Sobolev-class of orthogonal polynomials. This work is based on a joint work with my supervisor Adri Olde Daalhuis and has been published in [20].

Chapter 2

Exponentially accurate uniform asymptotic approximations for integrals and Bleistein 's method revisited

2.1 Introduction

In this chapter we obtain new uniform asymptotic approximations for integrals with an exponentially small remainder via a surprisingly simple method. We will also illustrate how these results can be used to obtain remainder estimates in the so-called Bleistein method. As an application of our results we show in the final section that

$$\begin{aligned}\frac{\Gamma(\lambda)}{\Gamma(\lambda+b)}{}_2F_1\left({a,b \atop \lambda+b};-z\right) \sim\ & \zeta^b e^{-a\zeta}\left(\frac{e^\zeta-1}{\zeta}\right)^{a+b-1} U\left(b,b-a+1,\lambda\zeta\right) \\ & +\zeta^b\left(1-e^{-a\zeta}\left(\frac{e^\zeta-1}{\zeta}\right)^{a+b-1}\right)U\left(b,b-a+2,\lambda\zeta\right),\end{aligned} \tag{2.1}$$

as $\lambda\to\infty$ in $|\mathrm{ph}\,\lambda|\leqslant\pi/2$ uniformly for large $|z|$. In this result, a and b are fixed complex constants and $\zeta=\ln\left(1+z^{-1}\right)$. For the notation of the hypergeometric function and the Kummer-U function see chapters 13 and 15 in [9]. Note that this result is a generalisation of the well-known limit

$$\lim_{c\to\infty} {}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};-cx\right) = x^{-b}U\left(b,b-a+1,x^{-1}\right). \tag{2.2}$$

See 6.8(1) in [11].

In [42] it was indicated how to obtain (2.1). The author even gave the integral representation for the left-hand-side of (2.1) that we will use. It seemed that all we had to do is just apply the Bleistein method. That is exactly what we will do in §2.4. However, when we tried to show that the remainder was of the required order we encountered a new problem: the approximant $U(a,b,z)$ has a relative complicated behaviour near $z=0$ as can be seen in §13.2(iii) in [27]. The usual methods did not work. Our new method is based on expanding the remainder in a new series with an exponentially small remainder. After deriving that method we realised that it can also be applied to the original integral itself. In that way we obtain a uniform asymptotic approximation with an exponentially small remainder. Note that the new approximations are not uniform asymptotic expansions. The expansions do not have an asymptotic property, but we will give estimates for the terms.

In chapter 1, we considered the integral of the form

$$\int_{\mathcal{C}} e^{\lambda p(t,\zeta)} q(t,\zeta) G(t)\, dt. \tag{2.3}$$

For these integrals the critical points are saddle points of p, the singularities of p and q, and possibly the endpoints of the contour of integration. We assumed that the position of these critical points depends on a parameter ζ, that they coalesce at the origin when $\zeta=0$, and that $G(t)$ is analytic for at least $|t|\leqslant 1$. The functions p and q are chosen in such a way that the integral without the G is a good approximant. To obtain an asymptotic expansion for $|\lambda|\to\infty$ that holds uniformly for ζ near the origin all the relevant critical points should contribute. The Bleistein method is a multipoint expansion of $G(t)$ about the relevant critical points combined with a special integration by parts. For examples see [5], [12], [13], [30], [37], [44], [48], [49], and chapter vii in [47].

The Bleistein method is a very subtle process. We will show that one can also obtain a uniform asymptotic approximation by substituting the truncated Taylor series $G(t) = \sum_{m=0}^{M-1} g_m t^m + t^M S(t)$, where we insist that the number of terms depends on λ, that is, $M-\gamma|\lambda| \in [0,1)$, for some positive constant γ. In this way we obtain an approximation with a remainder of the form (2.3), in which $G(t)$ is replaced by $t^M S(t)$. By taking M large enough the integral representation for this remainder will have a dominant saddle point outside the region of coalescence, and simple estimates show that the remainder is exponentially small compared with the first approximants.

This chapter is organised as follows. In §2.2 we first illustrate the main ideas for probably the best known example: 2 coalescing saddles. We also give a few details on how one could obtain a uniform asymptotic approximation with an exponentially small remainder. The main class of integrals is introduced in §2.3. These are integrals of the form (2.3) with $p(t,\zeta) = t$, $q(t,\zeta) = t^{b-1}(1+t/\zeta)^{-a}$ and $\mathcal{C} = [0,\infty)$. Theorem 2.3.1 contains the uniform asymptotic approximation with an exponentially small remainder, and in Lemma 2.3.2 we translate this result into an order estimate for the integral. This order estimate is exactly what is needed in §2.4 in which we apply the Bleistein method to obtain the uniform asymptotic expansion for this class of integrals. Finally we apply these results in §2.5 to the hypergeometric function mentioned on the left-hand side of (2.1). We obtain the uniform asymptotic expansion in terms of Kummer-U functions, and illustrate the new uniform asymptotic approximation with exponentially small remainder.

2.2 Uniform asymptotics for integrals

In §1.3.4 in chapter 1, we illustrated the main steps in the process of obtaining uniform asymptotic expansions via the Bleistein method with probably the best known example: 2 coalescing saddles. In the case that one encounters an integral in which two coalescing saddles dominate the asymptotics one first converts the integral to its canonical form. We considered the integral

$$f(\lambda,\zeta) = \frac{1}{2\pi i}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} e^{\lambda\left(\frac{1}{3}t^3-\zeta t\right)} G(t)\,dt. \tag{2.4}$$

Often in the literature, the function $G(t)$ also depends on ζ. Without loss of generality and for simplicity of presentation, we write $G(t)$ instead of $G(t,\zeta)$. Furthermore we consider only $\lambda > 0$ and assume that the function $G(t)$ is analytic for $|t| < 1+\varepsilon$, where ε is a positive constant, and also that $G(t)$ is analytic and bounded along the path of integration. We are interested in the case that $\lambda \to \infty$, for ζ in some neighbourhood of the origin, say $|\zeta| \leqslant \frac{1}{4}$. Note that the phase function has saddle points at $t = \pm\sqrt{\zeta}$. In the case that $\zeta > 0$ there exists a steepest descent path just passing through one saddle point, and in the case that $\zeta < 0$ the steepest descent path will pass through both saddle points. See Figure 4.1.

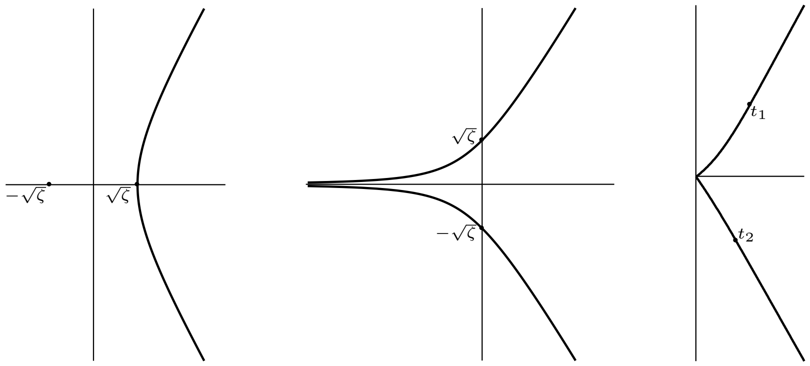

Figure 2.1: The left figure is for $\zeta > 0$. In that case the steepest descent path of integration in (2.4) passes through only one saddle point. The figure in the middle is for the case $\zeta < 0$. The right figure is the steepest descent curve for (2.8) in the case $\zeta = \frac{1}{4}e^{\pi i/4}$.

It follows that in the case that we want an asymptotic approximation for $\lambda \to \infty$ that holds for ζ close to the origin, then both saddle points should contribute. One option would be to use a multipoint expansion like the one that is discussed in [6], but we used the Bleistein method explained in §1.3.4 in Chapter 1. For more details see [30].

Let us investigate what would happen when we use some ideas from exponential asymptotics, and expand

$$G(t) = \sum_{m=0}^{M-1} g_m t^m + t^M S(t), \quad \text{with} \quad S(t) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{G(\tau)\tau^{-M}}{\tau - t}\, d\tau, \tag{2.5}$$

where for the moment $\mathcal{C}$ is a contour that encircles t and the origin once. We obtain

$$f(\lambda, \zeta) = \sum_{m=0}^{M-1} g_m u_m(\lambda, \zeta) + \tilde{R}_M(\lambda, \zeta), \tag{2.6}$$

with

$$u_m(\lambda, \zeta) = \frac{1}{2\pi i}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} e^{\lambda\left(\frac{1}{3}t^3 - \zeta t\right)} t^m\, dt, \tag{2.7}$$

and

$$\tilde{R}_M(\lambda,\zeta) = \frac{1}{2\pi i}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} e^{\lambda\left(\frac{1}{3}t^3-\zeta t\right)} t^M S(t)\, dt. \tag{2.8}$$

By taking a suitable contour for $\mathcal{C}$ we can guarantee that there exists a constant K such that $|S(t)| \leqslant K$ for all t on our path of integration. We also take $M = \lambda/2 + \rho$ with $\rho \in [0,1)$. Hence, the phase function in (2.8) becomes $p(t) = \frac{1}{3}t^3 - \zeta t + \frac{1}{2}\ln t$. Since $p'(t) = t^2 - \zeta + \frac{1}{2t}$ this phase function has three saddle points, say at $t = t_1, t_2, t_3$, and since we restrict $|\zeta| \leqslant \frac{1}{4}$ we can choose them such that $\operatorname{ph} t_1 \in (0, \frac{\pi}{2})$, $\operatorname{ph} t_2 \in (-\frac{\pi}{2}, 0)$ and $\Re(t_3) < 0$. Note that now two saddle points are active. In the lower half plane the steepest descent path emanates from $\infty e^{-\pi i/3}$, passes through t_2 and ends at the origin, and in the upper half plane it starts at the origin, passes through t_1 and ends at $\infty e^{\pi i/3}$. See Figure. 4.1.

The main difference between the integrals in (2.4) and (2.8) is the factor t^M. For the integral in (2.8) the saddle points are located at $t = t_1, t_2$ and we have $|t_j| < 1$, $j = 1, 2$. It follows that $\tilde{R}_M(\lambda,\zeta)$ is clearly exponentially small compared with $f(\lambda,\zeta)$. Hence, the finite sum in (2.6) is an asymptotic approximation for $f(\lambda,\zeta)$ with an exponentially small error. Via the saddle point method and using the identity $\frac{1}{3}t_j^3 = \frac{1}{3}\zeta t_j - \frac{1}{6}$ we obtain the estimate

$$\tilde{R}_M(\lambda,\zeta) = \frac{e^{-\lambda/6}}{\sqrt{\lambda}}\mathcal{O}\left(\left|e^{-\frac{2}{3}\lambda\zeta t_1} t_1^{\lambda}\right| + \left|e^{-\frac{2}{3}\lambda\zeta t_2} t_2^{\lambda}\right|\right), \quad \text{as } \lambda \to \infty. \tag{2.9}$$

We do not claim that expansion (2.6) has an asymptotic property. Let us investigate this expansion. We identify $u_0 = \lambda^{-1/3}\mathrm{Ai}\left(\zeta\lambda^{2/3}\right)$ and $u_1 = -\lambda^{-2/3}\mathrm{Ai}'\left(\zeta\lambda^{2/3}\right)$. Via integration by parts we obtain the recurrence relation

$$u_{m+2} - \zeta u_m + \frac{m}{\lambda}u_{m-1} = 0, \quad m = 0, 1, 2, \cdots, M-2, \tag{2.10}$$

where in the case $m = 0$ the third term vanishes. Hence, it is possible to express the u_m in terms of the first two:

$$u_m = a_m u_0 + b_m u_1, \tag{2.11}$$

where $a_0 = b_1 = 1$, $a_1 = b_0 = b_2 = 0$, $a_2 = \zeta$ and the a_m and b_m satisfy recurrence relation (2.10). The a_m and b_m are polynomials in ζ and $1/\lambda$. Since we assume $|\zeta| \leqslant \frac{1}{4}$

and take $m \leqslant M-2$, that is, $m/\lambda \leqslant \frac{1}{2}$, it is easy to show via induction that

$$|a_m| \leqslant (3/4)^{m/3}, \quad |b_m| \leqslant (3/4)^{(m-1)/3} \quad \text{for} \quad 0 \leqslant m \leqslant M. \tag{2.12}$$

Hence, we can rewrite (2.6) as

$$f(\lambda,\zeta) = \lambda^{-1/3}\mathrm{Ai}\left(\zeta\lambda^{2/3}\right)\sum_{m=0}^{M-1} g_m a_m - \lambda^{-2/3}\mathrm{Ai}'\left(\zeta\lambda^{2/3}\right)\sum_{m=0}^{M-1} g_m b_m + \tilde{R}_M(\lambda,\zeta). \tag{2.13}$$

Note that since we assume that $G(t)$ is analytic for $|t| < 1+\varepsilon$ it follows that the g_m are bounded. Thus the sums in (2.13) are clearly bounded, and it follows that there exists a constant K such that

$$|f(\lambda,\zeta)| \leqslant K\left(\lambda^{-1/3}\left|\mathrm{Ai}\left(\zeta\lambda^{2/3}\right)\right| + \lambda^{-2/3}\left|\mathrm{Ai}'\left(\zeta\lambda^{2/3}\right)\right|\right) \tag{2.14}$$

for say $|\zeta| \leqslant \frac{1}{4}$ and $\lambda > \lambda_0$. Hence, this is an alternative way of obtaining order estimates for these integrals. To obtain (1.80) one extra integration by parts was needed, whereas for the new method we expand in many terms.

Note that the expansions (1.78) and (2.13) are very similar. The two series in (1.78) clearly have an asymptotic property, whereas the two series in (2.13) behave like convergent geometric series. Often the calculation of the Taylor coefficients g_m is much easier than the calculation of the p_n and q_n. The big advantage of (1.78) is that even taking $N=1$ gives a one term approximation for the left-hand side, whereas in (2.13) we have to take many terms.

2.3 Main example: an exponentially small accurate uniform asymptotic approximation

For $a, b \in \mathbb{C}$ fixed parameters, we consider the following integral

$$f_{a,b}(\lambda,\zeta) = \frac{1}{\Gamma(b)}\int_0^\infty \frac{t^{b-1}e^{-\lambda t}}{(1+t/\zeta)^a} G(t)\, dt, \tag{2.15}$$

where $\Re(\lambda) > 0, \Re(b) > 0$ and $|\mathrm{ph}\,(\zeta)| < \pi$. The integrand in (2.15) has possible branch points at the origin and at $t = -\zeta$, which coalesce in the case that $\zeta = 0$. Let, again, ε be a positive constant. We assume that $G(t)$ is analytic in a domain $\mathcal{D}$, which is the union of the disc $|t| \leqslant 1+\varepsilon$ and the sector $|\mathrm{ph}\, t| \leqslant \theta_0$, where $\theta_0 = \frac{1}{8}\pi + \varepsilon$, which is chosen arbitrarily, to allow the singularities in the complex t plane but not along the real axis. See Figure 2.2. We also assume that $G(t)$ is of at most polynomial growth as $|t| \to \infty$ in $\mathcal{D}$.

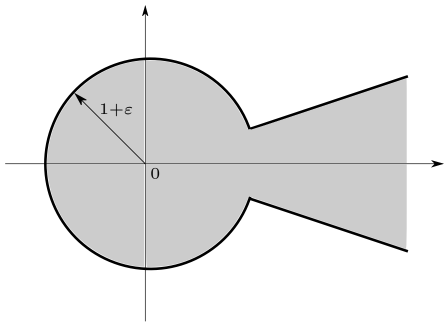

Figure 2.2: The shaded region $\mathcal{D}$ shows the area where $G(t)$ is analytic.

It is well-known that to obtain an asymptotic expansion for an integral of the form (2.15) for $|\lambda| \to \infty$ that is supposed to hold uniformly for ζ close to the origin, both the endpoint $t = 0$ and the branch point $t = -\zeta$ should contribute. The Bleistein method will do that for us and will give us the uniform asymptotic approximation in terms of the Kummer U-function. We will show the details in §2.4. However, the proof of that result is not straightforward, and one new step is needed in which we use the lemma mentioned at the end of this section. The surprising fact is that we can also obtain a uniform asymptotic approximation by just focussing on the endpoint $t = 0$ and let the number of terms depend on $|\lambda|$.

The main theorem of this chapter is:

Theorem 2.3.1. *Let $\Re(b) > 0$ and $G(t)$ analytic in region $\mathcal{D}$ as shown in Figure 2.2 and of at most polynomial growth as $|t| \to \infty$ in $\mathcal{D}$. Let $M = \cos(3\pi/8)|\lambda| + \gamma_1$, with $\gamma_1 \in [0, 1)$. Then*

$$\begin{aligned} f_{a,b}(\lambda,\zeta) \quad = \quad & \zeta^b U(b, b-a+1, \lambda\zeta) \sum_{m=0}^{M-1} g_m \alpha_m \\ & + b\zeta^{b+1} U(b+1, b-a+2, \lambda\zeta) \sum_{m=0}^{M-1} g_m \beta_m + R_M(\lambda,\zeta), \end{aligned} \tag{2.16}$$

with

$$|R_M(\lambda,\zeta)| = e^{-|\lambda|\cos(3\pi/8)}\mathcal{O}\left(|\lambda|^{-1/2}\right), \tag{2.17}$$

as $\lambda\to\infty$ *in* $|\mathrm{ph}\,\lambda|\leqslant\pi/2$ *uniformly for* $|\zeta|\leqslant 1/4$ *and* $|\mathrm{ph}\,\zeta|\leqslant\pi$.

Here $g_m, m=0,1,2,\cdots$, are the Taylor coeffcients of $G(t)$ about $t=0$ and they are bounded. We define $\alpha_0=1, \alpha_1=0, \beta_0=0, \beta_1=1$ and the other α_m, β_m satisfy the recurrence relation,

$$\lambda u_{m+2} = (b-a+1+m-\lambda\zeta)u_{m+1} + \zeta(b+m)u_m. \tag{2.18}$$

Hence, α_m, β_m are polynomials in a, b, ζ, and $1/\lambda$. Below we show that in the case that we take $|\lambda|$ large enough we have the bounds $|\alpha_m|\leqslant P^m$, $|\beta_m|\leqslant P^{m-1}$ for $m=0,1,\cdots,M$, with $P=0.9101397\cdots$.

Let us start with the contour of integration in (2.15). In the case that $\mathrm{ph}\,(\lambda)\neq 0$ we would like to rotate the contour to the line $\mathrm{ph}\,(t)=-\mathrm{ph}\,(\lambda)$. There are two possible restrictions:

(1) Since we assume that $G(t)$ is analytic in $\mathcal{D}$ we will, for $|t|\geqslant 1/2$, always take $|\mathrm{ph}\,t|\leqslant\pi/8$. To be more precise, we take for $|t|\geqslant 1/2$ the straight-line $\mathrm{ph}\,t=\theta$, with

$$\theta = \begin{cases} -\mathrm{ph}\,\lambda, & \text{if } |\mathrm{ph}\,\lambda|<\pi/8, \\ \pi/8, & \text{if } -\pi/2\leqslant\mathrm{ph}\,\lambda\leqslant-\pi/8, \\ -\pi/8, & \text{if } \pi/8\leqslant\mathrm{ph}\,\lambda\leqslant\pi/2. \end{cases} \tag{2.19}$$

(2) The integrand has a branch-point at $t=-\zeta$, and we will allow this branch-point to approach the positive real axis. Hence, we have to allow for indents in our new contour of integration $\mathcal{C}_1$. The worst case is illustrated in Figure 2.3, which is of the case $\mathrm{ph}\,(\lambda)<0$ and $\mathrm{ph}\,(\zeta)=-\pi$, that is, the branch-point has approached the positive real axis from the upper half plane. The indent will be a circular arc, in which we choose the radius to be $\delta|\zeta|$, where we choose $\delta=1/10$.

We use (2.5) in (2.15) and obtain

$$f_{a,b}(\lambda,\zeta) = \sum_{m=0}^{M-1} g_m u_m(\lambda,\zeta) + R_M(\lambda,\zeta), \tag{2.20}$$

where

$$u_m(\lambda,\zeta)=\frac{1}{\Gamma(b)}\int_{\mathcal{C}_1}\frac{e^{-\lambda t}t^{b+m-1}}{(1+t/\zeta)^a}\,dt, \tag{2.21}$$

and

$$R_M(\lambda,\zeta)=\frac{1}{2\pi i\Gamma(b)}\int_{\mathcal{C}_1}\int_{\mathcal{C}_2}\frac{e^{-\lambda t}t^{b+M-1}}{(1+t/\zeta)^a}\frac{G(\tau)\tau^{-M}}{\tau-t}\,d\tau\,dt. \tag{2.22}$$

where contour $\mathcal{C}_2$ encircles contour $\mathcal{C}_1$. For $|t|>1$ the distance between the two contours is ε, and contour $\mathcal{C}_2$ contains also a big part of the unit cycle. See Figure 2.3.

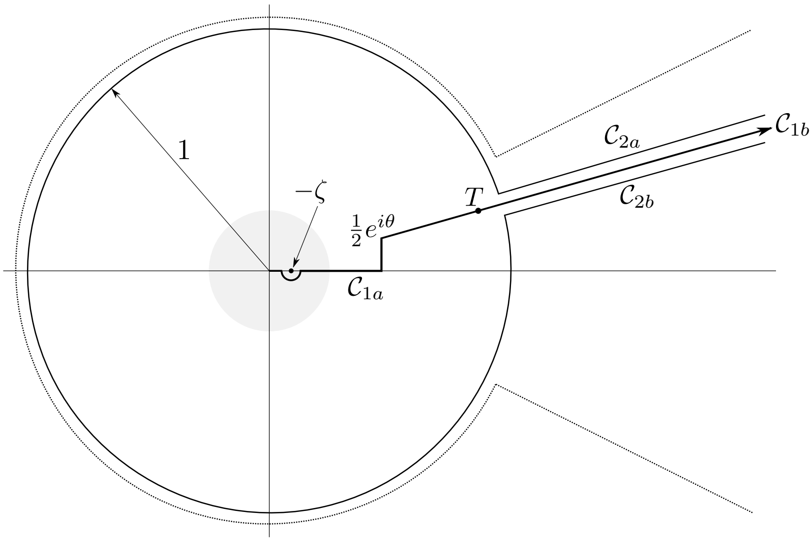

Figure 2.3: The contours $\mathcal{C}_1$ and $\mathcal{C}_2$. The contour $\mathcal{C}_1$ shows the worst case scenario. In simpler cases $\mathcal{C}_{1a}\cup\mathcal{C}_{1b}$ could just be the straight line $\operatorname{ph} t=\theta$. The shaded region is the cut-disc $\{\zeta : |\zeta|\leqslant 1/4 \text{ and } |\operatorname{ph}\zeta|<\pi\}$.

We will use the standard notation for the Kummer-U function given in chapter 13 of [9] and also integral representation (13.4.4) in that reference. We identify,

$$u_0(\lambda,\zeta)=\zeta^b U(b,b-a+1,\lambda\zeta),\qquad u_1(\lambda,\zeta)=b\zeta^{b+1}U(b+1,b-a+2,\lambda\zeta). \tag{2.23}$$

The reader can verify via integrating by parts that the functions $u_m(\lambda,\zeta)$ satisfy recurrence relation (2.18). It follows that $u_m(\lambda,\zeta)$ can be written as a linear combination of $u_0(\lambda,\zeta)$ and $u_1(\lambda,\zeta)$, i.e.,

$$u_m(\lambda,\zeta)=\alpha_m u_0(\lambda,\zeta)+\beta_m u_1(\lambda,\zeta), \tag{2.24}$$

where the α_m, β_m are defined above. Let us write $M = \gamma_0|\lambda| + \gamma_1$, such that $\gamma_1 \in [0, 1)$. We are going to show that there is a $P \in (0, 1)$ such that $|\alpha_m| \leqslant P^m$ for $m = 0, 1, \cdots, M$. This inequality clearly holds for $m = 0, 1$. Suppose that $|\alpha_j| \leqslant P^j$ for $j = m, m+1$, then we obtain from recurrence relation (2.18)

$$|\alpha_{m+2}| \leqslant \left(\left|\frac{b-a+1}{\lambda}\right| + \left|\frac{m}{\lambda}\right| + |\zeta| + |\zeta|\left(\left|\frac{b}{\lambda}\right| + \left|\frac{m}{\lambda}\right|\right)\right) P^m. \tag{2.25}$$

Recall that we already assume that $|\zeta| \leqslant \frac{1}{4}$. We will also take $|\lambda|$ large enough to guarantee that $|b-a+1| + |\zeta b| \leqslant |\lambda|/10$. Then we have

$$|\alpha_{m+2}| \leqslant \left(\tfrac{7}{20} + \tfrac{5}{4}\left|\frac{m}{\lambda}\right|\right) P^m \leqslant \left(\tfrac{7}{20} + \tfrac{5}{4}\gamma_0\right) P^m, \tag{2.26}$$

where we use for the second inequality that $m \leqslant M - 2 < \gamma_0|\lambda|$. Thus we want to take P such that $P^2 = \frac{7}{20} + \frac{5}{4}\gamma_0$. We can take for example $\gamma_0 = \cos(3\pi/8)$, then $P = 0.9101397\cdots$. With these choices for the constant we also have $|\beta_m| \leqslant P^{m-1}$ for $m = 0, 1, \cdots, M$.

Now we want to show that the remainder term defined in (2.22) is exponentially small. Since the growth of $G(\tau)$ is at most polynomial for large τ, we can assume that for M large enough, $G(\tau)\tau^{-M} = \mathcal{O}\left(\tau^{-2}\right)$ as $\tau \to \infty$. With our choice for contour $\mathcal{C}_2$ it follows that

$$\int_{\mathcal{C}_2} \frac{G(\tau)\tau^{-M}}{\tau - t}\, d\tau = \mathcal{O}(1), \tag{2.27}$$

as $\lambda \to \infty$, uniformly with respect to $t \in \mathcal{C}_1$. Hence, there is a constant K such that

$$|R_M(\lambda, \zeta)| \leqslant K \int_{\mathcal{C}_1} \left|\frac{e^{-\lambda t} t^{b+M-1}}{(1+t/\zeta)^a}\right| |dt|. \tag{2.28}$$

We divide the contour $\mathcal{C}_1 = \mathcal{C}_{1a} \cup \mathcal{C}_{1b}$, where $\mathcal{C}_{1a}$ is the finite path which starts at the origin and ends at $t = \frac{1}{2}e^{i\theta}$, and $\mathcal{C}_{1b}$ is the path $t = re^{i\theta}$, $r \in [\frac{1}{2}, \infty)$. For the finite path we will derive below that there is a constant K_1 such that

$$K \int_{\mathcal{C}_{1a}} \left|\frac{e^{-\lambda t} t^{b+M-1}}{(1+t/\zeta)^a}\right| |dt| \leqslant K_1 e^{|\lambda\zeta|\delta} 2^{-M}. \tag{2.29}$$

Along the contour $\mathcal{C}_{1a}$, we note that $\left|(1+t/\zeta)^{-a}\right|$ is bounded. Typically the exponential in the integrand is bounded by 1. However, for example, in the case that $\mathrm{ph}\,\lambda = -\pi/2$, and $-\zeta$ has approached the positive real axis from above, we have to make a small indent in the contour of integration, an arc with radius $\delta\,|\zeta|$. See again Figure 2.3. In that case the exponential along $\mathcal{C}_{1a}$ is bounded by $e^{|\lambda\zeta|\delta}$. Note that for all t along $\mathcal{C}_{1a}$ we have $|t| \leqslant 1/2$.

For the path $\mathcal{C}_{1b}$, we let $t = e^{i\theta}\tau$ where $|\theta| \leqslant \frac{\pi}{8}$. We obtain

$$K\int_{\mathcal{C}_{1b}} \left|\frac{e^{-\lambda t}t^{b+M-1}}{(1+t/\zeta)^a}\right| |dt| \;\leqslant\; K_2\int_{1/2}^{\infty} \left|e^{-\lambda\tau e^{i\theta}}\tau^{b+M-1}\right| d\tau \qquad (2.30)$$
$$= \; K_2\int_{1/2}^{\infty} e^{-|\lambda|\left(\tau\cos(\theta+\operatorname{ph}\lambda)-\gamma_0\ln\tau\right)}\tau^{\Re(b)-1+\gamma_1}\,d\tau,$$

where, again, we have used $M = \gamma_0|\lambda| + \gamma_1$. Here the phase function $f(\tau) = \tau\cos(\theta + \operatorname{ph}\lambda) - \gamma_0\ln\tau$ has a minimum at $\tau = \frac{\gamma_0}{\cos(\theta+\operatorname{ph}\lambda)}$. We take $T = \frac{\gamma_0}{\cos(\theta+\operatorname{ph}\lambda)}e^{i\theta}$. Applying the saddle point method (see §2.4(iv) in [34]) shows us that

$$K\int_{\mathcal{C}_{1b}} \left|\frac{e^{-\lambda t}t^{b+M-1}}{(1+t/\zeta)^a}\right| |dt| = e^{-|\lambda T|\cos(\theta+\operatorname{ph}\lambda)}\,|T|^M\,\mathcal{O}\left(|\lambda|^{-1/2}\right), \qquad (2.31)$$

as $|\lambda| \to \infty$. When we take, as before, $\gamma_0 = \cos(3\pi/8)$ and use the fact that with our choice for θ (see (2.19)), we have $|T| \leqslant 1$, we can write this result as

$$K\int_{\mathcal{C}_{1b}} \left|\frac{e^{-\lambda t}t^{b+M-1}}{(1+t/\zeta)^a}\right| |dt| = e^{-|\lambda|\cos(3\pi/8)}\mathcal{O}\left(|\lambda|^{-1/2}\right), \qquad (2.32)$$

as $|\lambda| \to \infty$.

Combining (2.28), (2.29) and (2.32) we have shown that (2.17) holds.

Lemma 2.3.2. *Let $\Re(b) > 0$ and $G(t)$ analytic in region $\mathcal{D}$ and of at most polynomial growth as $|t| \to \infty$ in $\mathcal{D}$. Then*

$$f_{a,b}(\lambda,\zeta) = \mathcal{O}\left(\left|\zeta^b U(b, b-a+1, \lambda\zeta)\right| + \left|b\zeta^{b+1}U(b+1, b-a+2, \lambda\zeta)\right|\right). \qquad (2.33)$$

as $\lambda \to \infty$ in $|\operatorname{ph}\lambda| \leqslant \pi/2$ uniformly for $|\zeta| \leqslant 1/4$ and $|\operatorname{ph}\zeta| \leqslant \pi$.

The confluent hypergeometric function $U(a, b, z)$ is one of the standard solution of

$$z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0 \qquad (2.34)$$

which is a second order linear differential equation. Let for a moment $w(\lambda) = \zeta^b U(b, b -$

$a+1, \lambda\zeta)$. Then it follows from (13.3.22) in [27] that $w'(\lambda) = -b\zeta^{b+1}U(b+1, b-a+2, \lambda\zeta)$. Since $w(\lambda)$ satisfies a second order linear differential equation, it follows that the two terms on the right-hand side of (2.33) cannot vanish at the same time. Hence, we can absorb the exponentially small remainder $R_M(\lambda, \zeta)$ into the right-hand side of (2.33). Since $G(t)$ is analytic in $\mathcal{D}$ it follows that there is a constant K such that $|g_m| \leqslant K$ for all m. Hence, $\left|\sum_{m=0}^{M-1} g_m\alpha_m\right| \leqslant K\left(1-P^M\right)/(1-P)$, and similarly $\left|\sum_{m=0}^{M-1} g_m\beta_m\right|$ is also bounded. Thus the right-hand side of (2.16) can be absorbed into the right-hand side of (2.33).

2.4 Main example: a uniform asymptotic expansion

Now we will obtain the uniform asymptotic expansion for (2.15) via Bleistein's method. We take $G_0(t) = G(t)$ and define $H_n(t)$, $G_{n+1}(t)$, $n = 0, 1, 2, \cdots$, by writing

$$G_n(t) = p_n + q_n(t+\zeta) + t(t+\zeta)H_n(t), \tag{2.35}$$

and

$$G_{n+1}(t) = (t+\zeta)^a\, t^{1-b}\frac{d}{dt}\left(\frac{t^b}{(t+\zeta)^{a-1}}H_n(t)\right), \tag{2.36}$$

with p_n, q_n following from the substitution of $t = 0$ and $t = -\zeta$:

$$p_n = G_n(-\zeta), \qquad\qquad q_n = \frac{G_n(0) - G_n(-\zeta)}{\zeta}. \tag{2.37}$$

Theorem 2.4.1. *Let $\Re(b) > 0$ and $G(t)$ analytic in region $\mathcal{D}$ and of at most polynomial growth as $|t| \to \infty$ in $\mathcal{D}$. Then*

$$\begin{aligned} f_{a,b}(\lambda,\zeta) &= \zeta^b U\left(b, b-a+1, \lambda\zeta\right)\sum_{j=0}^{n-1}\frac{p_j}{\lambda^j} + \zeta^{b+1}U\left(b, b-a+2, \lambda\zeta\right)\sum_{j=0}^{n-1}\frac{q_j}{\lambda^j} \\ &\quad +\lambda^{-n}R_n(\lambda,\zeta), \end{aligned} \tag{2.38}$$

with

$$R_n(\lambda,\zeta) = \mathcal{O}\left(\left|\zeta^b U\left(b, b-a+1, \lambda\zeta\right)\right| + \left|\zeta^{b+1}U\left(b, b-a+2, \lambda\zeta\right)\right|\right). \tag{2.39}$$

as $\lambda \to \infty$ in $|\mathrm{ph}\,\lambda| \leqslant \pi/2$ uniformly for $|\zeta| \leqslant 1/4$ and $|\mathrm{ph}\,\zeta| \leqslant \pi$.

To obtain the uniform asymptotic expansion we apply the Bleistein method, i.e. we substitute (2.35) (with $n = 0$) in (2.15):

$$f_{a,b}(\lambda,\zeta) = \frac{1}{\Gamma(b)} \int_0^\infty \frac{t^{b-1}e^{-\lambda t}}{(1+t/\zeta)^a} \left(p_0 + q_0(t+\zeta) + t(t+\zeta)H_0(t)\right) dt. \tag{2.40}$$

Integration by parts and using (2.36) results into

$$\begin{aligned} f_{a,b}(\lambda,\zeta) &= p_0\zeta^b U\left(b, b-a+1, \lambda\zeta\right) + q_0\zeta^{b+1} U\left(b, b-a+2, \lambda\zeta\right) \\ &+ \frac{1}{\lambda\Gamma(b)} \int_0^\infty \frac{e^{-\lambda t}t^{b-1}}{(1+t/\zeta)^a} G_1(t)\, dt. \end{aligned} \tag{2.41}$$

Since the integral on the right-hand side of (2.41) is of the same form as (2.15), we can repeat this process and obtain (2.38) with

$$R_n(\lambda,\zeta) = \frac{1}{\Gamma(b)} \int_{\mathcal{C}_1} \frac{e^{-\lambda t}t^{b-1}}{(1+t/\zeta)^a} G_n(t) dt. \tag{2.42}$$

We note that the process defined in (2.35) and (2.36) does not introduce new singularities for $G_n(t)$. Thus the $G_n(t)$ are analytic in $\mathcal{D}$ and of at most polynomial growth as $|t| \to \infty$ in $\mathcal{D}$. (For more details see Theorem 4.2 in Chapter 1 in [31]).

At this stage one would split the proof into 2 cases. The case $|\lambda\zeta|$ is bounded is much harder than usual since the behaviour of $U(\alpha, \beta, x)$ is complicated for x near the origin. See §13.2(iii) in [27]. Hence, it is not easy to just use integral representation (2.42) and obtain order estimate (2.39). Even in the much simpler case of two coalescing saddles, discussed in §2.2, one extra integration by parts was needed to obtain the required order estimate. Here that one extra step would not be sufficient. However, using many extra steps works. The result is Lemma 2.3.2. Note that the integral in (2.42) is the same as the one in the definition of $f_{a,b}(\lambda,\zeta)$, with $G(t)$ replaced by $G_n(t)$. Hence, we can use Lemma 2.3.2 and obtain

$$R_n(\lambda,\zeta) = \mathcal{O}\left(\left|\zeta^b U(b, b-a+1, \lambda\zeta)\right| + \left|b\zeta^{b+1} U(b+1, b-a+2, \lambda\zeta)\right|\right). \tag{2.43}$$

as $\lambda \to \infty$ in $|\mathrm{ph}\,\lambda| \leqslant \pi/2$ uniformly for $|\zeta| \leqslant 1/4$ and $|\mathrm{ph}\,\zeta| \leqslant \pi$. This is not exactly (2.39), but since recurrence relation (13.3.9) in [27] can be presented as

$$bU(b+1, b-a+2, \lambda\zeta) = U(b, b-a+2, \lambda\zeta) - U(b, b-a+1, \lambda\zeta), \tag{2.44}$$

and order estimate (2.39) follows.

2.5 The main application

The large λ asymptotics of the hypergeometric function ${}_2F_1\left(\begin{matrix}a,b\\ \lambda\end{matrix};z\right)$ in the case that a, b are fixed complex parameters and z is a bounded variable, is well understood. For large parts of the bounded complex z-plane the Gauss series itself

$$ {}_2F_1\left(\begin{matrix}a,b\\ \lambda\end{matrix};z\right) \sim \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(\lambda)_n\,n!} z^n, \tag{2.45} $$

is an asymptotic expansion. For more details and restrictions see [45]. In this section we will consider unbounded $|z|$.

The following substitution and expansion were suggested in [42] but the details were omitted. Here we give the detail derivation and consider

$$ {}_2F_1\left(\begin{matrix}a,b\\ \lambda+b\end{matrix};-z\right) = \frac{\Gamma(\lambda+b)}{\Gamma(\lambda)\Gamma(b)} \int_0^1 \frac{\tau^{b-1}\,(1-\tau)^{\lambda-1}}{(1+\tau z)^a}\,d\tau, \tag{2.46} $$

where $\Re(\lambda)>0$ and $\Re(b)>0$ and $|\mathrm{ph}\,(1+z)|<\pi$. Substituting $1-\tau=e^{-t}$ in (4.5), we obtain

$$ {}_2F_1\left(\begin{matrix}a,b\\ \lambda+b\end{matrix};-z\right) = \frac{\Gamma(\lambda+b)}{\Gamma(\lambda)\Gamma(b)} \int_0^\infty \frac{\left(1-e^{-t}\right)^{b-1}}{(1+z-ze^{-t})^a} e^{-\lambda t} dt, \tag{2.47} $$

which we write as

$$ {}_2F_1\left(\begin{matrix}a,b\\ \lambda+b\end{matrix};-z\right) = \frac{\Gamma(\lambda+b)}{\Gamma(\lambda)\Gamma(b)} \int_0^\infty \frac{t^{b-1}e^{-\lambda t}}{(1+t/\zeta)^a} G(t)\,dt, \tag{2.48} $$

where

$$ G(t) = \left(\frac{1-e^{-t}}{t}\right)^{b-1} \left(\frac{1+t/\zeta}{1+z-ze^{-t}}\right)^{a}, \tag{2.49} $$

and

$$ \zeta = \ln\left(1+\frac{1}{z}\right). \tag{2.50} $$

To guarantee that $|\zeta|\leqslant 1/4$ we take $|z|\geqslant \left(e^{1/4}-1\right)^{-1}$ and keep on assuming that $|\mathrm{ph}\,(1+z)|<\pi$. Note that when $z\to\infty$ then $\zeta\to 0$.

The singularities of $G(t)$ are at the points

$$t = 2k\pi i - \zeta, \qquad t = 2k\pi i, \qquad k = \pm 1, \pm 2, \pm 3 \cdots . \tag{2.51}$$

As $|t| \to \infty$ in sector $\mathcal{D}$ we have $G(t) = \mathcal{O}\left(t^{a-b+1}\right)$. Hence, we can use Theorem 3.13:

$$\begin{aligned} \frac{\Gamma(\lambda)}{\Gamma(\lambda+b)} {}_2F_1\left(\begin{matrix} a, b \\ \lambda+b \end{matrix}; -z\right) &= \zeta^b U\left(b, b-a+1, \lambda\zeta\right) \sum_{j=0}^{n-1} \frac{p_j}{\lambda^j} \\ &\quad + \zeta^{b+1} U\left(b, b-a+2, \lambda\zeta\right) \sum_{j=0}^{n-1} \frac{q_j}{\lambda^j} + \lambda^{-n} R_n(\lambda, \zeta), \end{aligned} \tag{2.52}$$

with order estimate (2.39). The coefficients p_n and q_n can be found using the process defined in (2.35) and (2.36). For the coefficients p_0 and q_0 we need

$$G(0) = 1, \qquad G(-\zeta) = e^{-a\zeta} \left(\frac{e^{\zeta} - 1}{\zeta}\right)^{a+b-1}. \tag{2.53}$$

Taking $n = 1$ in (2.52) gives us the approximation (2.1).

As a final numerical example we apply Theorem 2.3.1 to the hypergeometric function (2.48):

$$\begin{aligned} \frac{\Gamma(\lambda)}{\Gamma(\lambda+b)} {}_2F_1\left(\begin{matrix} a, b \\ \lambda+b \end{matrix}; -z\right) &\approx \zeta^b U(b, b-a+1, \lambda\zeta) \sum_{m=0}^{M-1} g_m \alpha_m \\ &\quad + b\zeta^{b+1} U(b+1, b-a+2, \lambda\zeta) \sum_{m=0}^{M-1} g_m \beta_m, \end{aligned} \tag{2.54}$$

where the Taylor coefficients g_m of $G(t)$ about $t = 0$ can be computed via (2.49). However, this process is not straightforward. It is more convenient to consider the logarithmic derivative of $G(t)$:

$$\begin{aligned} \frac{G'(t)}{G(t)} &= \frac{(b-1)\left((t+1)e^{-t} - 1\right)}{t\left(1 - e^{-t}\right)} + \frac{a\left(1 + z - ze^{-t}\left(t + \zeta + 1\right)\right)}{(t+\zeta)\left(1 + z - ze^{-t}\right)} \\ &= \sum_{k=0}^{\infty} h_k t^k. \end{aligned} \tag{2.55}$$

The computation of the h_k is much easier and to compute the g_m we use $g_0 = 1$ and

$$g_{m+1} = \frac{1}{m+1} \sum_{k=0}^{m} g_k h_{m-k}, \qquad m = 0, 1, \cdots, M-2. \tag{2.56}$$

If we take $a = 7/5$, $b = 23/10$, $\lambda = 10i$ and $z = 4$. Then $M = 3$, $\zeta = 0.22314355$ and the 'exact' value of the left-hand side of (2.54) is $-0.0028078130 - 0.0011238098i$, whereas the right-hand side gives us the approximation $-0.0028078267 - 0.0011237993i$. Thus the relative error is 5.71×10^{-6}. In the case that we take $\lambda = 20i$ we need $M = 7$ and the relative error is 6.46×10^{-12}.

Chapter 3

Uniform asymptotic expansions for hypergeometric functions with large parameters

3.1 Introduction

In this chapter, we study the asymptotics of the following hypergeometric function

$$ {}_2F_1\left(\begin{matrix} e_1\lambda + a, e_2\lambda + b \\ e_3\lambda + c \end{matrix}; \omega\right), \tag{3.1} $$

where $e_j = 0, \pm 1$, $j = 1, 2, 3$ as $|\lambda| \to \infty$ and obtain the asymptotic approximations near all critical values of ω as $\lambda \to \infty$. We mainly take $\omega = z$ or $\omega = -z$. The latter case has the advantage in the presentation of the results that the hypergeometric function has a branch-cut along part of the negative z plane. To find the asymptotic behaviour of the hypergeometric function, when one or more parameters are large is complicated. To obtain large ω-regions of validity one needs to use uniform asymptotic expansions. Several contributions already exists in the literature for the uniform asymptotics of Gauss hypergeometric functions (see [31], [25], [26], [29] [19] and [42]) which are valid for large values of λ. See §1.4 in Chapter 1

We showed in §1.4 in Chapter 1 that how the 27 cases can be reduced into 5 main cases as shown in Table 1.4 using the linear transformations Eqs.(1.33)-(1.35) and the connection formulas for Kummer's 24 solutions of the hypergeometric differential equations (see (15.10(ii)) in [28]).

For most of the cases discussed in this chapter, we first give the uniform asymptotic approximations using Bleistein's method, since these are valid in large regions of the variable. The other tool that we will use is the saddle point method for contour integrals.

In this chapter, we will consider δ to be a small positive real number.

To obtain the simplified form of the asymptotic approximation of the hypergeometric function, one needs to use the asymptotic expansion of the gamma function (see (15.11.3) in [28]) i.e.

$$\frac{\Gamma(\lambda+a)}{\Gamma(\lambda+b)} \sim \lambda^{a-b}\sum_{k=0}^{\infty}\frac{G_k(a,b)}{\lambda^k}, \tag{3.2}$$

as $\lambda\to\infty$ and $|\operatorname{ph}\lambda|<\pi$. Also

$$G_0(a,b)=1, \qquad G_1(a,b)=\frac{1}{2}(a-b)(a+b-1). \tag{3.3}$$

In the case, when the gamma functions contains 2λ, then to get the form as in (3.2), one can use the duplication formula (5.5.5) in [4], i.e.

$$\Gamma(2\lambda)=\pi^{-1/2}2^{2\lambda-1}\Gamma(\lambda)\Gamma(\lambda+1/2). \tag{3.4}$$

Another useful identity which we use is as follows

$$\frac{\Gamma(b)\Gamma(1-b)-e^{-a\pi i}\Gamma(a+b)\Gamma(1-a-b)}{\Gamma(b)\Gamma(1-b)}=e^{b\pi i}\frac{\Gamma(a+b)\Gamma(1-a-b)}{\Gamma(a)\Gamma(1-a)}, \tag{3.5}$$

for fixed $a,b,c\in\mathbb{C}$. This result is a direct consequence of the reflection formula (5.5.3) in [4].

This chapter is organised as follows. Referring to Table 1.4, in §§ 3.2,3.3,3.4 we will deal with cases 1, 2, 3, respectively, and in §3.5 we will deal with cases 4 and 5. The so-called Bleistein method is used several times in this paper, but only in case 3.3 do we give full details, and in that case we also provide an integral representation for the coefficients in these uniform asymptotic expansions. One reason to give full details for this case is that in the integral representation four critical points coalesce, but we note that only three of them contribute to the asymptotics. This seems to be a new observation.

3.2 Case 1: (0,0,±1) and (1,0,0)

We start this chapter by considering the case (0,0,1) for all the critical values of z. The case when z is large is discussed recently in [21], where the authors obtain the uniform asymptotic approximation of this case as $\lambda \to \infty$ in $|\mathrm{ph}\,\lambda| \leqslant \pi/2$ uniformly valid for large z. We give that result in §§3.2.1 in terms of the confluent hypergeometric U-function, better known as the Kummer U-function (see Chapter 13 in [9]).

In [14] the authors provide a uniform asymptotic expansion of the associated Legendre function

$$P_\nu^m(z) = \left(\frac{z-1}{z+1}\right)^{m/2} {}_2F_1\left(\begin{matrix}-\nu, \nu+1\\ m+1\end{matrix}; \frac{1-z}{2}\right), \tag{3.6}$$

for large m and fixed ν. Expressing the right-hand side of (3.6) in an integral representation, they obtain the uniform asymptotic approximation in terms of K-Bessel functions which is a special case of the result given in [21], since one has (13.6.10) in [27] i.e.

$$U\left(\nu+\tfrac{1}{2}, 2\nu+1, 2z\right) = \frac{1}{\sqrt{\pi}} e^z \left(2z\right)^{-\nu} K_\nu(z). \tag{3.7}$$

The large c-asymptotics seems to be the simplest case that we discuss in this chapter. Copying the details of §15.12(ii) of [28] we let δ being a small positive constant. Also let a, b, z be fixed, and at least one of the following conditions be satisfied

(a) a and/or $b \in \{0, -1, -2, \cdots\}$.

(b) $\Re z < \frac{1}{2}$ and $|c+n| \geqslant \delta$ for all $n \in \{0, 1, 2, \cdots\}$.

(c) $\Re z = \frac{1}{2}$ and $|\mathrm{ph}\,c| \leqslant \pi - \delta$.

(d) $\Re z > \frac{1}{2}$ and $\alpha_- - \frac{1}{2}\pi + \delta \leqslant \mathrm{ph}\,c \leqslant \alpha_+ + \frac{1}{2}\pi - \delta$, where

$$\alpha_\pm = \arctan\left(\frac{\mathrm{ph}\,z - \mathrm{ph}\,(1-z) \mp \pi}{\ln|1-z^{-1}|}\right), \tag{3.8}$$

with z restricted so that $\pm\alpha_\pm \in [0, \frac{1}{2}\pi)$.

Then for fixed $m \in \{0, 1, 2, \cdots\}$, we have

$${}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; z\right) = \sum_{s=0}^{m-1} \frac{(a)_s\,(b)_s}{(c)_s\, s!} z^s + \mathcal{O}\left(c^{-m}\right), \qquad |c| \to \infty. \tag{3.9}$$

We will use this result to deal with the case of bounded z. Since we will make use of linear transformations it makes sense to replace c by $c+\lambda$. For the case of Condition (d) the sector of validity is very complicated, and we will use much simpler sectors for our results.

3.2.1 Case: (0,0,1)

bounded z **:** $\Re z \leqslant \frac{1}{2}$

Using the Condition (b) in the previous section and the result in (3.9), we have

$$ {}_2F_1\left(\begin{matrix} a,b \\ \lambda+c \end{matrix};z\right) = \sum_{s=0}^{m-1} \frac{(a)_s\,(b)_s}{(\lambda+c)_s\,s!} z^s + \mathcal{O}\left(\lambda^{-m}\right), \tag{3.10} $$

as $\lambda \to \infty$, $|\mathrm{ph}\,\lambda| \leqslant \pi - \delta$.

bounded z **:** $\Re z \geqslant \frac{1}{2}$

Using the condition (d) above we have $\alpha_- \in \left[-\frac{\pi}{2}, 0\right]$ and $\alpha_+ \in \left[0, \frac{\pi}{2}\right]$, and again (3.10) holds as $\lambda \to \infty$, but now in the sector $|\mathrm{ph}\,\lambda| \leqslant \frac{\pi}{2} - \delta$.

For $\delta \leqslant |\mathrm{ph}\,\lambda| \leqslant \pi - \delta$, we use the linear transformation 15.10.21 in [28]:

$$ \begin{aligned} {}_2F_1\left(\begin{matrix} a,b \\ \lambda+c \end{matrix};z\right) \quad = \quad & \frac{\Gamma(\lambda+c)\Gamma(\lambda+c-a-b)}{\Gamma(\lambda+c-b)\Gamma(\lambda+c-a)} {}_2F_1\left(\begin{matrix} a,b \\ a+b-c-\lambda+1 \end{matrix};1-z\right) \\ & + \frac{\Gamma(\lambda+c)\Gamma(a+b-c-\lambda)}{\Gamma(a)\Gamma(b)} \frac{(1-z)^{\lambda+c-a-b}}{z^{\lambda+c-1}} {}_2F_1\left(\begin{matrix} 1-a,1-b \\ \lambda-a-b+c+1 \end{matrix};1-z\right). \end{aligned} \tag{3.11} $$

Since we assume $\Re z \geqslant \frac{1}{2}$, we have $\Re(1-z) \leqslant \frac{1}{2}$ and thus we can use the result in (3.10), that is

$$
\begin{aligned}
&{}_2F_1\left({a,b \atop c+\lambda};z\right)\\
&\quad=\frac{\Gamma(\lambda+c)\Gamma(\lambda+c-a-b)}{\Gamma(\lambda+c-b)\Gamma(\lambda+c-a)}\left(\sum_{s=0}^{m-1}\frac{(a)_s\,(b)_s}{(a+b-c-\lambda+1)_s\,s!}(1-z)^s+\mathcal{O}\left(\lambda^{-m}\right)\right)\\
&\qquad+\frac{\Gamma(\lambda+c)\Gamma(a+b-c-\lambda)}{\Gamma(a)\Gamma(b)}\frac{(1-z)^{\lambda+c-a-b}}{z^{\lambda+c-1}}\\
&\qquad\qquad\times\left(\sum_{s=0}^{m-1}\frac{(1-a)_s\,(1-b)_s}{(\lambda-a-b+c+1)_s\,s!}(1-z)^s+\mathcal{O}\left(\lambda^{-m}\right)\right),
\end{aligned}
\tag{3.12}
$$

as $\lambda\to\infty$. When we express an asymptotic approximation in terms of a sum of two divergent series, there is always the risk that the two divergent series could annihilate each other. Note that the first term on the right-hand side of (3.12) is of order $1+abz\lambda^{-1}+\mathcal{O}\left(\lambda^{-2}\right)$, and the asymptotic approximation of the second term involves the factors $\left(z^{-1}-1\right)^{\lambda}\lambda^{a+b-1}/\sin(\pi(a+b-c-\lambda))$. Hence, there is no risk of annihilation.

Large z

The complete uniform asymptotic approximation for large z is given in [21] and taking only the dominant terms we have

$$
\begin{aligned}
\frac{\Gamma(\lambda)}{\Gamma(\lambda+b)}{}_2F_1\left({a,b \atop \lambda+b};-z\right)\ \sim\ &\zeta^b e^{-a\zeta}\left(\frac{e^{\zeta}-1}{\zeta}\right)^{a+b-1}U\left(b,b-a+1,\lambda\zeta\right)\\
&+\zeta^b\left(1-e^{-a\zeta}\left(\frac{e^{\zeta}-1}{\zeta}\right)^{a+b-1}\right)U\left(b,b-a+2,\lambda\zeta\right),
\end{aligned}
\tag{3.13}
$$

as $\lambda\to\infty$ in $|\mathrm{ph}\,\lambda|\leqslant\pi/2$ uniformly for $|\zeta|\leqslant\frac{1}{4}$ and $|\mathrm{ph}\,\zeta|\leqslant\pi$ where $\zeta=\ln\left(1+z^{-1}\right)$. In this result, a and b are fixed complex constants. The U function is the confluent hypergeometric function (see Chapter 13 in [9]).

We note that our asymptotic approximations of the case (0,0,1) for bounded z cover the sector $|\mathrm{ph}\,\lambda|\leqslant\pi-\delta$, while when z is large then our approximation is only valid for $|\mathrm{ph}\,\lambda|\leqslant\frac{\pi}{2}$.

3.2.2 Case: (0,0,-1)

This case can be transformed to the previous case (0,0,1) by using the linear transformation (15.10.17) in [28]:

$$\begin{aligned}
&\frac{1}{\Gamma(c-\lambda)}{}_2F_1\left({a,b \atop c-\lambda};-z\right)\\
&\quad=\frac{\Gamma(\lambda+a-c+1)\Gamma(\lambda+b-c+1)}{\Gamma(\lambda+a+b-c+1)\Gamma(\lambda-c+1)\Gamma(c-\lambda)}{}_2F_1\left({a,b \atop \lambda+a+b-c+1};z+1\right)\\
&\qquad+\frac{\Gamma(\lambda+a-c+1)\Gamma(\lambda+b-c+1)\,(-z)^{\lambda-c+1}}{\Gamma(a)\Gamma(b)\Gamma(\lambda-c+2)\,(z+1)^{\lambda-c+a+b}}{}_2F_1\left({1-a,1-b \atop \lambda-c+2};-z\right).
\end{aligned} \tag{3.14}$$

Note that the function on the left-hand side has no poles in the finite complex λ plane.

z-bounded

When z is bounded, it is enough to consider $|\mathrm{ph}\,\lambda| \leqslant \frac{\pi}{2}-\delta$ since the other region of λ is covered in §3.2.1. Thus one can get the asymptotic approximation by combining the right-hand side of (3.14) with (3.10). Again, it is easy to verify that the two divergent series cannot annihilate each other.

Large z

For large z, we apply (3.13) to the right-hand side of (3.14) and obtain

$$\begin{aligned}
&\frac{1}{\Gamma(c-\lambda)}{}_2F_1\left({a,b \atop c-\lambda};-z\right)\sim\Gamma(\lambda+b-c+1)\zeta^{b}\\
&\times\Bigg(\frac{e^{\pm\pi i(c-\lambda)+(c-a-\lambda-1)\zeta}}{\Gamma(a)\Gamma(b)}\Big(-U\left(1-a,b-a+1,(\lambda+a-c+1)\,\zeta\right)\\
&\qquad+C(\zeta)U\left(1-a,b-a+2,(\lambda+a-c+1)\,\zeta\right)\Big)\\
&+\frac{e^{\pm b\pi i}\sin((c-\lambda)\pi)}{\pi}\Big(U\left(b,b-a+1,e^{\pm\pi i}\,(\lambda+a-c+1)\,\zeta\right)\\
&\qquad+bC(\zeta)U\left(b+1,b-a+2,e^{\pm\pi i}\,(\lambda+a-c+1)\,\zeta\right)\Big)\Bigg),
\end{aligned} \tag{3.15}$$

as $\lambda \to \infty$ in $|\mathrm{ph}\,\lambda| \leqslant \pi/2$ uniformly for $|\zeta| \leqslant \frac{1}{4}$, $|\mathrm{ph}\,\zeta| \leqslant \pi$ and $\mathrm{ph}\left((\lambda+a-c+1)\zeta\right) \lesseqgtr 0$, where $\zeta = \ln\left(1+z^{-1}\right)$ and

$$C(\zeta) = 1 - e^{a\zeta}\left(\frac{1-e^{-\zeta}}{\zeta}\right)^{a+b-1}. \tag{3.16}$$

Note that with the choice of the signs in the exponentials in (3.15) we guarantee that

$$\mathrm{ph}\,\left(e^{\pm\pi i}\left(\lambda+a-c+1\right)\zeta\right) \in [-\pi,\pi]. \tag{3.17}$$

3.2.3 Case: (1,0,0)

We start with the linear transformation (15.10.29) in [28]:

$$\begin{aligned} {}_2F_1\left({\lambda+a,b \atop c};-z\right) &= \frac{\Gamma(c)\Gamma(\lambda+a-c+1)}{\Gamma(\lambda+a+b-c+1)\Gamma(c-b)} z^{-b}{}_2F_1\left({b,b-c+1 \atop \lambda+a+b-c+1};1+\frac{1}{z}\right) \\ &\quad + \frac{\Gamma(c)\Gamma(\lambda+a-c+1)\,(-z)^{b-c}}{\Gamma(\lambda+a-b+1)\Gamma(b)\,(z+1)^{\lambda+a+b-c}}{}_2F_1\left({1-b,c-b \atop \lambda+a-b+1};-\frac{1}{z}\right). \end{aligned} \tag{3.18}$$

Note that the whole real line is a branch cut for this identity since the hypergeometric functions from left in (3.18) have branch cuts from $z=-1$ to $-\infty$, $z=0$ to ∞, and $z=-1$ to 0 respectively. For z bounded away from the origin and $|\mathrm{ph}\,\lambda| \leqslant \frac{1}{2}\pi - \delta$, we can replace in the right-hand side of (3.18) the two hypergeometric functions by their Gauss series (3.10). Again, it is easy to verify that the two divergent series cannot annihilate each other.

$|z+1| \leqslant 1$ and z bounded away from 0

Note that in this case we have $\Re\left(1+z^{-1}\right) \leqslant \frac{1}{2}$. Hence, in the linear transformation (15.10.21) in [28]

$$\begin{aligned} {}_2F_1\left({\lambda+a,b \atop c};-z\right) &= \frac{\Gamma(c)\Gamma(c-a-b-\lambda)}{\Gamma(c-b)\Gamma(c-a-\lambda)}\,(-z)^{-b}\,{}_2F_1\left({b,b-c+1 \atop \lambda+a+b-c+1};1+\frac{1}{z}\right) \\ &\quad + \frac{\Gamma(c)\Gamma(\lambda+a+b-c)\,(-z)^{b-c}}{\Gamma(\lambda+a)\Gamma(b)\,(z+1)^{\lambda+a+b-c}}{}_2F_1\left({1-b,c-b \atop c-a-b-\lambda+1};1+\frac{1}{z}\right), \end{aligned} \tag{3.19}$$

we can apply the Gauss series (3.10) to the hypergeometric functions on the right-hand side. We obtain asymptotic expansions that are valid for $\lambda \to \infty$ such that $\delta \leqslant |\mathrm{ph}\,\lambda| \leqslant \pi - \delta$.

$|z+1| \geqslant 1$ and z bounded away from 0

Now we have $\Re\left(-z^{-1}\right) \leqslant \frac{1}{2}$ and in the linear transformation (15.10.25) in [28]

$$\begin{aligned} {}_2F_1\left(\begin{matrix}\lambda+a, b\\ c\end{matrix}; -z\right) &= \frac{\Gamma(c)\Gamma(\lambda+a-b)}{\Gamma(\lambda+a)\Gamma(c-b)} z^{-b} {}_2F_1\left(\begin{matrix}b, b-c+1\\ b-a-\lambda+1\end{matrix}; -\frac{1}{z}\right) \\ &\quad + \frac{\Gamma(c)\Gamma(b-a-\lambda) z^{b-c}}{\Gamma(c-a-\lambda)\Gamma(b)\,(z+1)^{\lambda+a+b-c}} {}_2F_1\left(\begin{matrix}1-b, c-b\\ \lambda+a-b+1\end{matrix}; -\frac{1}{z}\right), \end{aligned} \tag{3.20}$$

we can apply the Gauss series (3.10) to the hypergeometric functions on the right-hand side, and obtain asymptotic expansions that are valid for $\lambda \to \infty$ such that $\delta \leqslant |\mathrm{ph}\,\lambda| \leqslant \pi - \delta$.

z close to 0

Now we take $\zeta = \ln(z+1)$ and assume that $|\zeta| \leqslant \frac{1}{4}$ and $|\mathrm{ph}\,\zeta| \leqslant \pi$. We use (3.13) for the hypergeometric functions appearing on the right-hand side of (3.18) and obtain as $\lambda \to \infty$

$$\begin{aligned} \frac{\Gamma(\lambda+a)}{\Gamma(\lambda+a-b+1)} {}_2F_1\left(\begin{matrix}1-b, c-b\\ \lambda+a-b+1\end{matrix}; -\frac{1}{z}\right) &\sim \frac{(z+1)^{b-c}}{\zeta^{c-b-1} z^{2b-c}} \Big(U\left(1-b, 2-c, (\lambda+a)\zeta\right) \\ &\quad - C(z) U\left(1-b, 3-c, (\lambda+a)\zeta\right)\Big), \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} &\frac{\Gamma(\lambda+a)}{\Gamma(\lambda+a+b-c+1)} {}_2F_1\left(\begin{matrix}b, b-c+1\\ \lambda+a+b-c+1\end{matrix}; 1+\frac{1}{z}\right) \\ &\quad \sim e^{\pm(c-b+1)\pi i}\Big(\zeta^{1-b}\,(z+1)^{c-b}\, z^{2b-c} U\left(b-c+1, 2-c, e^{\pm\pi i}(\lambda+a)\zeta\right) \\ &\qquad + \zeta^{b-c+1} C(z) U\left(b-c+1, 3-c, e^{\pm\pi i}(\lambda+a)\zeta\right)\Big), \end{aligned} \tag{3.22}$$

when $\mathrm{ph}\,((\lambda+a)\zeta) \lessgtr 0$, where

$$C(z) = 1 - (z+1)^{c-b}\left(\frac{z}{\zeta}\right)^{2b-c}. \tag{3.23}$$

Combining (3.21) and (3.22) and using the connection formulas given in 13.2(vii) in [28], we obtain

$$\begin{aligned}{}_2F_1\left({\lambda+a,b \atop c};-z\right) \sim{}& \frac{\Gamma(\lambda+a-c+1)\,(\lambda+a)^{c-1}}{\Gamma(\lambda+a)\,(z+1)^{\lambda+a}}\left(\frac{\zeta}{z}\right)^{b}\Big(M\left(c-b,c,(\lambda+a)\,\zeta\right)\\ &+\frac{c-1}{\zeta(\lambda+a)}C(z)M\left(c-b-1,c-1,(\lambda+a)\,\zeta\right)\Big),\end{aligned} \tag{3.24}$$

which can be simplified to

$$\begin{aligned}{}_2F_1\left({\lambda+a,b \atop c};-z\right) \sim{}& \frac{\Gamma(\lambda+a-c+1)\,(\lambda+a)^{c-1}\,\zeta^{b-1}}{\Gamma(\lambda+a+1)\,(z+1)^{\lambda+a}\,z^{b}}(c-1)\Bigg(M\left(c-b,c-1,(\lambda+a)\zeta\right)\\ &-(z+1)^{c-b}\left(\frac{z}{\zeta}\right)^{2b-c}M\left(c-b-1,c-1,(\lambda+a)\zeta\right)\Bigg),\end{aligned} \tag{3.25}$$

as $\lambda\to\infty$ and $|\mathrm{ph}\,\lambda| \leqslant \pi/2$.

3.3 Case 2: (1,-1,0)

The case discussed below in Theorem 3.3.1 is also discussed in Jones [19]. He started from a second order linear differential equation and his work is based on the ideas of Olver [31]. Our result differs slightly from Jones's result, but can be converted into his expansion. However, in §3.3.1 we derive a new integral representation for the coefficients appearing in the uniform asymptotic expansion, and we use the same techniques for the case discussed in Theorem 3.3.2.

By taking the linear transformation 15.10.17 in [28] we have

$$
{}_2F_1\left({\lambda+a,a-\lambda \atop c};\frac{1-z}{2}\right)=\frac{\Gamma(c)\Gamma(c-2a)}{\Gamma(c-a-\lambda)\Gamma(\lambda+c-a)}{}_2F_1\left({\lambda+a,a-\lambda \atop 2a+1-c};\frac{z+1}{2}\right)
+\frac{\Gamma(c)\Gamma(2a-c)}{\Gamma(\lambda+a)\Gamma(a-\lambda)}\left(\frac{1-z}{2}\right)^{1-c}\left(\frac{z+1}{2}\right)^{c-2a}{}_2F_1\left({\lambda-a+1,1-a-\lambda \atop c+1-2a};\frac{z+1}{2}\right), \tag{3.26}
$$

and it seems possible to express the case discussed in Theorem 3.3.2 in terms of the case discussed in Theorem 3.3.1. However, if we take for example $\lambda=i|\lambda|$, then we obtain from Theorem 3.3.2 the large λ approximation

$$
{}_2F_1\left({\lambda+a,a-\lambda \atop c};\frac{1-z}{2}\right)\sim e^{-\pi i(\lambda-c/2+1/4)-\xi\lambda}\frac{\Gamma(c)\Gamma(\lambda+a-c+1)\,(z+1)^{c/2-a-1/4}}{2^{1-a}\Gamma(\lambda+a)\sqrt{\lambda\pi}\,(1-z)^{c/2-1/4}}, \tag{3.27}
$$

and from Theorem 3.3.1 we obtain

$$
\frac{\Gamma(c)\Gamma(c-2a)}{\Gamma(c-a-\lambda)\Gamma(\lambda+c-a)}{}_2F_1\left({\lambda+a,a-\lambda \atop 2a-c+1};\frac{z+1}{2}\right)
\sim e^{-\pi i(\lambda-c/2+1/4)+\lambda\xi}\frac{\Gamma(c)\Gamma(\lambda+a-c+1)\,(z+1)^{c/2-a-1/4}}{2^{2-a}\Gamma(\lambda+a)\sqrt{\lambda\pi}\,(1-z)^{c/2-1/4}\sin\left(\pi(c-2a)\right)}, \tag{3.28}
$$

where ξ is defined in (3.34). From these results, it follows that the left-hand side of (3.26) is much smaller than the two terms on the right-hand side of (3.26). Hence, combining Jones's result with the linear transformation (3.26) would lead to a massive cancellation of the dominant terms. Due to these massive cancelations we cannot use linear transformation (3.26) and we need the result stated in Theorem 3.3.2.

Theorem 3.3.1. *For fixed $a,c\in\mathbb{C}$ and $|\mathrm{ph}\,(z-1)|<\pi$, we have*

$$
{}_2F_1\left({\lambda+a,a-\lambda \atop c};\frac{1-z}{2}\right)=\frac{\Gamma(c)\Gamma(\lambda+a-c+1)}{\Gamma(\lambda+a)}\left(\left(\frac{\zeta}{2}\right)^{1-c}I_{c-1}(\zeta\lambda)\sum_{j=0}^{n-1}\frac{a_j}{\lambda^j}
+\left(\frac{\zeta}{2}\right)^{-c}I_c(\zeta\lambda)\sum_{j=1}^{n-1}\frac{b_j}{\lambda^j}+\mathcal{O}(\Phi_n(\lambda,\zeta))\right), \tag{3.29}
$$

as $\lambda \to \infty$ *in* $|\mathrm{ph}\,\lambda| \leqslant \frac{\pi}{2}$, *where*

$$\zeta = \ln(z + \sqrt{z^2 - 1}). \tag{3.30}$$

The first two coefficients are

$$a_0 = 2^{a-c+\frac{1}{2}}\,(z+1)^{\frac{c}{2}-a-\frac{1}{4}}\,(z-1)^{\frac{1}{4}-\frac{c}{2}}\,\zeta^{c-\frac{1}{2}}, \qquad b_0 = 0. \tag{3.31}$$

The asymptotic sequence $\Phi_n(\lambda, \zeta)$ *is defined by*

$$\Phi_n(\lambda, \zeta) = \left|\zeta^{1-c}\lambda^{-n} I_{c-1}(\zeta\lambda)\right| + \left|\zeta^{-c}\lambda^{-n} I_c(\zeta\lambda)\right|. \tag{3.32}$$

Theorem 3.3.2. *For fixed* $a, c \in \mathbb{C}$, $|\mathrm{ph}\,(1-z)| < \pi$ *and* $|\mathrm{ph}\,(z+1)| < \pi$, *we have*

$$\begin{aligned}
{}_2F_1\left(\begin{matrix}\lambda+a, a-\lambda\\ c\end{matrix}; \frac{1-z}{2}\right) &= \frac{\Gamma(c)\Gamma(\lambda+a-c+1)}{\pi i \Gamma(\lambda+a)} \\
&\times\Bigg(\left(\frac{\xi}{2}\right)^{c-2a}\left(e^{(\lambda-a)\pi i} K_{2a-c}(-\xi\lambda) - e^{(c-a-\lambda)\pi i} K_{2a-c}(\xi\lambda)\right)\sum_{j=0}^{n-1}\frac{c_j}{\lambda^j} \\
&\quad + \left(\frac{\xi}{2}\right)^{c-2a-1}\left(e^{(\lambda-a)\pi i} K_{2a-c+1}(-\xi\lambda) - e^{(c-a-\lambda)\pi i} K_{2a-c+1}(\xi\lambda)\right)\sum_{j=0}^{n-1}\frac{d_j}{\lambda^j} \\
&\quad + \mathcal{O}(\Phi_n(\lambda, \xi))\Bigg),
\end{aligned} \tag{3.33}$$

as $\lambda \to \infty$ *in* $|\mathrm{ph}\,\lambda| \leqslant \frac{\pi}{2}$, *where*

$$\xi = \ln\left(-z - i\sqrt{1-z^2}\right). \tag{3.34}$$

The first two coefficients are

$$c_0 = 2^{c-a-\frac{1}{2}} e^{\pi i\left(a-\frac{c}{2}-\frac{3}{4}\right)}\left(\frac{z+1}{1-z}\right)^{\frac{c}{2}-\frac{1}{4}}(z+1)^{-a}\,\xi^{2a-c+\frac{1}{2}}, \qquad d_0 = 0. \tag{3.35}$$

The asymptotic sequence $\Phi_n(\lambda,\zeta)$ *is defined by*

$$\Phi_n(\lambda,\zeta) = \left|e^{(\lambda-a)\pi i}\xi^{c-2a}\lambda^{-n}K_{2a-c}(-\xi\lambda)\right| + \left|e^{(c-a-\lambda)\pi i}\xi^{c-2a}\lambda^{-n}K_{2a-c}(\xi\lambda)\right| \\ + \left|e^{(\lambda-a)\pi i}\xi^{c-2a-1}\lambda^{-n}K_{2a-c+1}(-\xi\lambda)\right| + \left|e^{(c-a-\lambda)\pi i}\xi^{c-2a-1}\lambda^{-n}K_{2a-c+1}(\xi\lambda)\right|. \tag{3.36}$$

3.3.1 The proofs

Before we give the proofs we first have to derive a convenient integral representation.

The integral representation

For the moment, we assume $z > 1$ and $\Re(\lambda + a) > 0$, $c - a - \lambda \neq 1, 2, 3, \cdots$. We combine Eqs. (1.33)-(1.34) with integral representation (15.6.3) in [28] and obtain

$${}_2F_1\left(\begin{matrix}\lambda+a, a-\lambda\\ c\end{matrix}; \frac{1-z}{2}\right) = \frac{e^{(\lambda+a-c)\pi i}\Gamma(c)\Gamma(\lambda+a-c+1)}{2\pi i\Gamma(\lambda+a)}\int_{\infty}^{(0+)} \frac{t^{c-a-\lambda-1}\,(t+1)^{a-c-\lambda}}{\left(t+\frac{z+1}{2}\right)^{a-\lambda}}\,dt. \tag{3.37}$$

Now we substitute $\tau = e^{-\pi i}t$ in the above integral representation

$${}_2F_1\left(\begin{matrix}\lambda+a, a-\lambda\\ c\end{matrix}; \frac{1-z}{2}\right) = \frac{\Gamma(c)\Gamma(\lambda+a-c+1)}{2\pi i\Gamma(\lambda+a)}\int_{-\infty}^{(0+)} \frac{\tau^{c-a-\lambda-1}\,(1-\tau)^{a-c-\lambda}}{\left(\frac{1}{2}+\frac{z}{2}-\tau\right)^{a-\lambda}}\,d\tau. \tag{3.38}$$

Here the path of integration starts at $e^{-\pi i}\infty$ encircles 0 once in the positive direction and returns to $e^{\pi i}\infty$. The points 1 and $\frac{z+1}{2}$ lie outside the contour of integration.

We write

$${}_2F_1\left(\begin{matrix}\lambda+a, a-\lambda\\ c\end{matrix}; \frac{1-z}{2}\right) = \frac{L}{2\pi i}\int_{-\infty}^{(0+)} e^{\lambda f(t)}g(t)\,dt, \tag{3.39}$$

where

$$f(t) = \ln\left(\frac{\frac{z+1}{2}-t}{1-t}\right) - \ln t, \qquad g(t) = \frac{t^{c-a-1}\,(1-t)^{a-c}}{\left(\frac{z+1}{2}-t\right)^{a}}, \tag{3.40}$$

and

$$L = \frac{\Gamma(c)\Gamma(\lambda+a-c+1)}{\Gamma(\lambda+a)}. \tag{3.41}$$

In the right-hand side of (3.40), we indicate that we choose for the phase function $f(t)$

branch cuts between $t = \frac{1+z}{2}$ and $t = 1$ and the negative real axis. The saddle points are

$$sp_{\pm} = \frac{z+1}{2} \pm \frac{1}{2}\sqrt{z^2 - 1}. \tag{3.42}$$

The branch points of the phase function are $t = 0$, $t = 1$ and $t = \frac{z+1}{2}$. The saddle points and two of the branch points coalesce when $z \to 1$ at $t = 1$. Although, four points coalesce with each other, one should see this as the coalescence of $sp_{\pm}$ with $t = \frac{z+1}{2}$, since $t = \frac{z+1}{2}$ is one of the end points of the steepest descent paths and $f(\frac{z+1}{2}) = -\infty$ and $f(1) = +\infty$. Hence, the value of f at these branch points is considerably different.

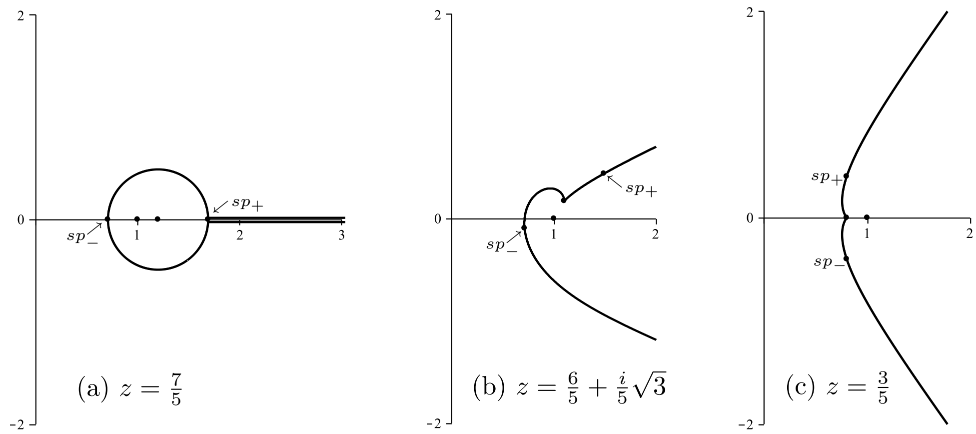

Figure 3.1: In (b) and (c), the contour of the integral in (3.39) starts at $t = \infty$ in the lower half plane passes the saddle point $t = sp_-$ and continues till $t = \frac{z+1}{2}$. Then after passing the saddle point $t = sp_+$ goes to ∞ in the upper half plane. Case (a), is the limiting case i.e. the contour emanates from $t = \infty$ and after passing through the saddle point $t = sp_+$ makes a loop passing through $t = sp_-$ to $t = sp_+$ and finally returns to ∞.

In Figure 3.1 we display the steepest descent contours in the case that λ is positive and for several complex values of z. According to (3.39) the paths should start at $-\infty$. However, since for large t we have $e^{\lambda f(t)} g(t) \sim t^{-\lambda-a-1}$, it follows that we can deform the contour in the (3.39) to the steepest descents contours that are displayed in Figure 3.1.

Theorem 3.3.1: uniform asymptotics

To obtain a uniform asymptotic expansion, we use the transformation[1] (see [40])

$$f(t) = \tau + \frac{\zeta^2}{4\tau} + \eta. \tag{3.43}$$

The saddle points $t = sp_\pm$ correspond to $\tau = \mp\zeta/2$. Thus,

$$f(sp_\pm) = \mp\zeta + \eta. \tag{3.44}$$

We obtain $\eta = 0$ and ζ is given in (3.30) i.e. $\zeta = \operatorname{arccosh}(z)$. In Figure 3.2 we show that the cut-plane $|\mathrm{ph}\,(z-1)| < \pi$ is mapped to half strip $\Re\zeta > 0$ and $|\Im\zeta| < \pi$.

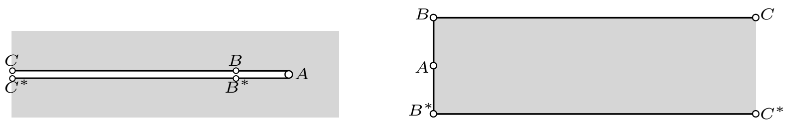

Figure 3.2: The $z \longleftrightarrow \zeta$ mapping, where in the z-plane (left), $A = 1$, $B = -1 + \varepsilon i$, $B^* = -1 - \varepsilon i$, $C = -\infty + \varepsilon i$, $C^* = \infty - \varepsilon i$, and in the ζ-plane (right), A=0, $B = \pi i$, $B^* = -\pi i$, $C = \pi i + \infty$, $C^* = -\pi i + \infty$.

To obtain the singular behaviour of the coalesced version of the integrand we make the following observation. If $z = 1$ then $\zeta = 0$ and thus the transformation (3.43) reduces to $-\ln(t) = \tau$. Thus as $\tau \to 0$ we have $t \sim 1$, $\frac{dt}{d\tau} \sim -1$ and $g(t) = t^{c-a-1}(1-t)^{-c} \sim \tau^{-c}$.

With the transformation (3.43), we obtain integral representation

$${}_2F_1\left(\begin{matrix}\lambda + a, a - \lambda\\ c\end{matrix}; \frac{1-z}{2}\right) = \frac{L}{2\pi i}\int_{-\infty}^{(0+)} e^{\lambda\left(\tau + \frac{\zeta^2}{4\tau}\right)} G_0(\tau)\tau^{-c}\, d\tau, \tag{3.45}$$

with

$$G_0(\tau) = g(t)\frac{dt}{d\tau}\tau^c. \tag{3.46}$$

[1]The most obvious transformation would be $f(t) = \ln(\zeta + \tau) - \ln\tau + \tau + \eta$, which will give an approximation in terms of confluent hypergeometric functions, which, in turn, can be approximated in terms of Bessel functions. With (3.43) we obtain an approximation in terms of Bessel functions in one step.

The following are the values which we will need to find $G_0\left(\mp\frac{\zeta}{2}\right)$

$$g(sp_-) = \frac{2^{a+1}(z+1)^{\frac{c}{2}-a}(z-1)^{-\frac{c}{2}}}{1+e^{-\zeta}}, \qquad g(sp_+) = e^{\pm\pi ic-\zeta} g(sp_-), \tag{3.47}$$

depending on $\Im z \gtrless 0$. Since

$$\left(\frac{dt}{d\tau}\right)_{\tau=\mp\frac{\zeta}{2}} = \pm\sqrt{\frac{4}{\zeta f''(sp_\pm)}} = \pm e^{\pm\zeta/2}\sqrt{\frac{\sinh\zeta}{\zeta}}, \tag{3.48}$$

which can be obtained from (3.43) via l'Hôpital's method, we obtain

$$G_0\left(\mp\zeta/2\right) = 2^{a-c+\frac{1}{2}}(z+1)^{\frac{c}{2}-a-\frac{1}{4}}(z-1)^{\frac{1}{4}-\frac{c}{2}}\zeta^{c-\frac{1}{2}}. \tag{3.49}$$

Note that $G_0\left(-\zeta/2\right) = G_0\left(\zeta/2\right)$. However, $G_0(\tau)$ is not an even function.

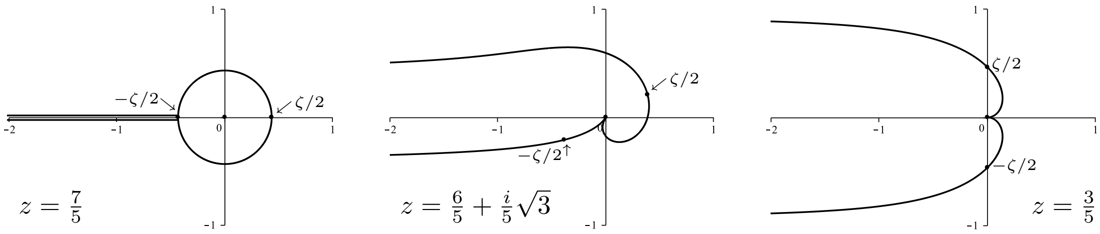

Figure 3.3: In (b) and (c), the contour in the integral in (3.45) starts at $\tau=-\infty$ in the lower half plane passes the saddle point $\tau=-\zeta/2$ and going through $\tau=0$ it passes the saddle point $\tau=\zeta/2$ returns to $-\infty$ in the upper half plane. Case (a) is the limiting case i.e. the contour emanates from $\tau=-\infty$ and after passing through the saddle point $\tau=-\zeta/2$ makes a loop passing through $\tau=\zeta/2$ to $\tau=-\zeta/2$ and then returns to $-\infty$.

So far, we have considered only $z>1$ and $\Re(a+\lambda)>0$, but now we analytically continue with $\lambda\in\mathbb{C}$, $z\in\mathbb{C}$ and $\zeta\in\mathbb{C}$. Thus, the integral (3.45) becomes

$${}_2F_1\left({\lambda+a, a-\lambda \atop c};\frac{1-z}{2}\right) = \frac{L}{2\pi i}\int_{\mathcal{C}} e^{\lambda\left(\tau+\frac{\zeta^2}{4\tau}\right)} G_0(\tau)\tau^{-c}\,d\tau, \tag{3.50}$$

where $\mathcal{C}$ is the steepest descent contour which starts at $e^{-(\mathrm{ph}\,(\lambda)+\pi)i}\infty$ and ends at $e^{-(\mathrm{ph}\,(\lambda)-\pi)i}\infty$.

To obtain the uniform asymptotic expansion, we use Bleistein's method [5], i.e. we substitute into (3.50)

$$G_n(\tau) = a_n + \frac{b_n}{\tau} + \left(1 - \frac{\zeta^2}{4\tau^2}\right) H_n(\tau), \tag{3.51}$$

and

$$G_{n+1}(\tau) = -\tau^c \frac{d}{d\tau}(\tau^{-c} H_n(\tau)), \tag{3.52}$$

with $n = 0$, the integral representation becomes

$${}_2F_1\left(\begin{matrix}\lambda + a, a - \lambda \\ c\end{matrix}; \frac{1-z}{2}\right) = \frac{L}{2\pi i}\int_{\mathcal{C}} e^{\lambda\left(\tau + \frac{\zeta^2}{4\tau}\right)}\left(a_0 + \frac{b_0}{\tau} + \left(1 - \frac{\zeta^2}{4\tau^2}\right) H_0(\tau)\right)\tau^{-c}\, d\tau. \tag{3.53}$$

We identify

$$\frac{1}{2\pi i}\int_{\mathcal{C}} e^{\lambda\left(\tau + \frac{\zeta^2}{4\tau}\right)}\frac{d\tau}{\tau^c} = 2^{c-1}\zeta^{1-c} I_{c-1}(\zeta\lambda). \tag{3.54}$$

Thus, by integration by parts and using the above two results, we obtain

$$\begin{aligned} {}_2F_1\left(\begin{matrix}\lambda + a, a - \lambda \\ c\end{matrix}; \frac{1-z}{2}\right) &= L\left(a_0\left(\frac{\zeta}{2}\right)^{1-c} I_{c-1}(\zeta\lambda) + b_0\left(\frac{\zeta}{2}\right)^{-c} I_c(\zeta\lambda)\right) \\ &\quad + \frac{L}{2\lambda\pi i}\int_{\mathcal{C}} e^{\lambda\left(\tau + \frac{\zeta^2}{4\tau}\right)} G_1(\tau)\,\frac{d\tau}{\tau^c}. \end{aligned} \tag{3.55}$$

Since the integrals in (3.50) and (3.55) are similar, we can repeat this process and obtain

$$\begin{aligned} {}_2F_1\left(\begin{matrix}\lambda + a, a - \lambda \\ c\end{matrix}; \frac{1-z}{2}\right) &= \frac{\Gamma(c)\Gamma(\lambda + a - c + 1)}{\Gamma(\lambda + a)}\left(\left(\frac{\zeta}{2}\right)^{1-c} I_{c-1}(\zeta\lambda)\sum_{j=0}^{n-1}\frac{a_j}{\lambda^j}\right. \\ &\quad \left. + \left(\frac{\zeta}{2}\right)^{-c} I_c(\zeta\lambda)\sum_{j=1}^{n-1}\frac{b_j}{\lambda^j} + \frac{\lambda^{-n}}{2\pi i}\int_{\mathcal{C}} e^{\lambda\left(\tau + \frac{\zeta^2}{4\tau}\right)} G_n(\tau)\,\frac{d\tau}{\tau^c}\right). \end{aligned} \tag{3.56}$$

The functions $G_n(\tau)$, $n = 1, 2, 3, \cdots$, are obtained from $G_0(\tau)$ via (3.51) and (3.52). This process does not introduce new singularities, thus the function $G_n(\tau)$ has the same singularities as $G_0(\tau)$, as well as the exponential behavior at infinity of both the functions are similar.

It follows from (3.51) that

$$a_n = \frac{G_n\left(\frac{\zeta}{2}\right) + G_n\left(-\frac{\zeta}{2}\right)}{2}, \qquad b_n = \frac{\zeta\left(G_n\left(\frac{\zeta}{2}\right) - G_n\left(-\frac{\zeta}{2}\right)\right)}{4}. \tag{3.57}$$

Using (3.49), we obtain

$$a_0 = 2^{a+\frac{1}{2}-c}\,(z+1)^{\frac{c}{2}-\frac{1}{4}-a}\,(z-1)^{\frac{1}{2}-\frac{c}{2}}\,\zeta^{c-\frac{1}{2}}, \qquad b_0 = 0. \tag{3.58}$$

To prove that the $\mathcal{O}$-term (3.32) holds, we need to show that

$$\int_{\mathcal{C}} e^{\lambda(\tau+\frac{\zeta^2}{4\tau})} G_n(\tau)\,\frac{d\tau}{\tau^c} = \mathcal{O}(\Phi_0(\lambda,\zeta)), \tag{3.59}$$

as $\lambda \to \infty$. Hence, we need to find the asymptotics of $\Phi_0(\lambda,\zeta)$ as $\lambda \to \infty$.

As $x \to \infty$ in the sector $-\frac{\pi}{2} < \pm\mathrm{ph}\, x < \frac{3\pi}{2}$ and c fixed, we have (see 10.40.5 in [33])

$$I_{c-1}(x) \sim \frac{e^x}{\sqrt{2\pi x}} \pm i e^{\pm(c-1)\pi i}\frac{e^{-x}}{\sqrt{2\pi x}}. \tag{3.60}$$

Using these results in (3.32) for $n = 0$, we obtain

$$\Phi_0(\lambda,\zeta) \sim \left|\zeta^{1-c}\left(\frac{e^{\zeta\lambda}}{\sqrt{2\pi\zeta\lambda}} \pm i e^{\pm(c-1)\pi i}\frac{e^{-\zeta\lambda}}{\sqrt{2\pi\zeta\lambda}}\right)\right| + \left|\zeta^{-c}\left(\frac{e^{\zeta\lambda}}{\sqrt{2\pi\zeta\lambda}} \pm i e^{\pm c\pi i}\frac{e^{-\zeta\lambda}}{\sqrt{2\pi\zeta\lambda}}\right)\right|, \tag{3.61}$$

as $|\zeta\lambda| \to \infty$ in $-\frac{\pi}{2} < \pm\mathrm{ph}\,(\zeta\lambda) < \frac{3\pi}{2}$. We note that the two terms in (3.61) cannot vanish at the same time, so the right-hand side of (3.61) is always non-zero.

If $|\zeta\lambda| \gg 1$, then the contributions from the saddle points at $\tau = \pm\zeta/2$ to the integral (3.59) will be,

$$\int_{\mathcal{C}} e^{\lambda(\tau+\frac{\zeta^2}{4\tau})} G_n(\tau)\,\frac{d\tau}{\tau^c} \sim -i2^{c-\frac{1}{2}} e^{\zeta\lambda}\frac{G_n(\zeta/2)}{\sqrt{2\pi\lambda}}\zeta^{\frac{1}{2}-c} + 2^{c-\frac{1}{2}} e^{-\zeta\lambda} e^{\pm\pi i(c-1)}\frac{G_n(-\zeta/2)}{\sqrt{2\pi\lambda}}\zeta^{\frac{1}{2}-c}, \tag{3.62}$$

as $|\lambda| \to \infty$, where the sign $\pm$ in the exponential indicates the direction of the contour. The first term on the right-hand side of (3.62) corresponds to the saddle point at $\tau = \zeta/2$ and the second to the saddle point at $\tau = -\zeta/2$.

For this case, i.e. $|\zeta\lambda| \gg 1$, by comparing (3.62) with (3.61), shows that (3.59) holds.

Now if $\zeta = \frac{x}{\lambda}$, where $x \in \mathbb{C}$ is bounded, we have

$$\Phi_0(\lambda, \zeta) = \left| \left(\frac{\lambda}{x}\right)^{c-1} I_{c-1}(x) \right| + \left| \left(\frac{\lambda}{x}\right)^{c} I_{c}(x) \right|. \tag{3.63}$$

From the theory of differential equations, we can say that the zeros of the two terms in (3.63) do not coincide with each other. We use the substitution $\lambda\tau = t$ in (3.59) and obtain

$$\int_{\mathcal{C}} e^{\lambda\left(\tau + \frac{\zeta^2}{4\tau}\right)} G_n(\tau)\, \frac{d\tau}{\tau^c} \quad = \quad \lambda^{c-1} \int_{\mathcal{C}'} e^{t + \frac{x^2}{4t}} G_n\left(\frac{t}{\lambda}\right) \frac{dt}{t^c} = \mathcal{O}\left(\lambda^{c-1}\right), \tag{3.64}$$

where $\mathcal{C}$ is mapped onto $\mathcal{C}'$ in the t-plane. By comparing (3.64) with (3.63), we can see that

$$\frac{1}{\lambda^{1-c}} \int_{\mathcal{C}'} e^{t + \frac{x^2}{4t}} G_n\left(\frac{t}{\lambda}\right) \frac{dt}{t^c} = \mathcal{O}\left(\Phi_0(\lambda, \zeta)\right). \tag{3.65}$$

Theorem 3.3.2: uniform asymptotics

We start with $-1 < z < 1$ and $\Re(a + \lambda) > 0$, $c - a - \lambda \neq 1, 2, 3, \cdots$. In integral representation (3.37), we take for the contour in the upper half plane, a contour $\mathcal{C}_+$ that starts at $t = \infty$ and ends at $t = -\frac{z+1}{2}$. For the contour in the lower half plane, we first use $t = \tilde{t}e^{2\pi i}$ and in the lower half $\tilde{t}$ plane, let $\mathcal{C}_-$ be the contour that starts at $\tilde{t} = -\frac{z+1}{2}$ and ends at $\tilde{t} = \infty$. Writing t for $\tilde{t}$, we have

$${}_2F_1\left(\begin{matrix} \lambda + a, a - \lambda \\ c \end{matrix}; \frac{1-z}{2}\right) = L_1 \int_{\mathcal{C}_+} e^{\lambda f(t)} g(t)\, dt + L_2 \int_{\mathcal{C}_-} e^{\lambda f(t)} g(t)\, dt, \tag{3.66}$$

where

$$f(t) = \ln\left(1 + \frac{z+1}{2t}\right) - \ln(1+t), \qquad g(t) = \frac{t^{c-a-1}(1+t)^{a-c}}{\left(\frac{z+1}{2} + t\right)^{a}}, \tag{3.67}$$

and

$$L_+ = \frac{e^{(\lambda+a-c)\pi i}\Gamma(c)\Gamma(\lambda+a-c+1)}{2\pi i \Gamma(\lambda+a)}, \qquad L_- = \frac{e^{(c-a-\lambda)\pi i}\Gamma(c)\Gamma(\lambda+a-c+1)}{2\pi i \Gamma(\lambda+a)}. \tag{3.68}$$

We choose the branch cuts of the phase function $f(t)$ between the points $t = -\frac{z+1}{2}$ and $t = 0$ and the half line $t < -1$. The saddle points are

$$sp_{\pm} = \pm i\frac{\sqrt{1-z^2}}{2} - \frac{z+1}{2}. \tag{3.69}$$

The saddle points and two of the branch points coalesce when $z \to -1$.

To obtain the uniform asymptotic expansion, we use the transformation

$$f(t) = -\tau - \frac{\xi^2}{4\tau} + \gamma. \tag{3.70}$$

The saddle points of the left-hand side of this transformation should correspond to those of the right hand side, that are $\tau = \pm\xi/2$. Thus

$$f(sp_{\pm}) = \mp\xi + \gamma. \tag{3.71}$$

We obtain $\gamma = 0$ and ξ is given in (3.34).

Now if $z = -1$, then $\xi = 0$ and the transformation (3.70) reduces to $-\ln(1+t) \approx -\tau$. Thus as $\tau \to 0$ we have $t \sim \tau$, $\frac{dt}{d\tau} \sim 1$ and $g(t) = t^{c-2a-1}(t+1)^{a-c} \sim \tau^{c-2a-1}$.

With the transformation (3.70), we obtain the integral representation

$$\begin{aligned} {}_2F_1\left(\begin{matrix}\lambda+a, a-\lambda \\ c\end{matrix}; \frac{1-z}{2}\right) &= L_+ \int_{\mathcal{C}_+} e^{-\lambda\left(\tau+\frac{\xi^2}{4\tau}\right)} G_0(\tau)\tau^{c-2a-1}\, d\tau \\ &\quad + L_- \int_{\mathcal{C}_-} e^{-\lambda\left(\tau+\frac{\xi^2}{4\tau}\right)} G_0(\tau)\tau^{c-2a-1}\, d\tau, \end{aligned} \tag{3.72}$$

with

$$G_0(\tau) = g(t)\frac{dt}{d\tau}\tau^{2a-c+1}. \tag{3.73}$$

We will need

$$\left(\frac{dt}{d\tau}\right)_{\tau=\pm\frac{\xi}{2}} = \sqrt{\frac{-\xi^2}{2\tau^3 f''(sp_{\pm})}} = \sqrt{\pm\frac{1+ze^{\mp\xi}}{\xi}}, \tag{3.74}$$

which can be computed from (3.70) via l'Hôpital's method. It follows that

$$G_0(\pm\xi/2) = -2^{c-a-\frac{1}{2}} e^{\pi i\left(a-\frac{c}{2}-\frac{3}{4}\right)} \left(\frac{z+1}{1-z}\right)^{\frac{c}{2}-\frac{1}{4}} (z+1)^{-a}\, \xi^{2a-c+\frac{1}{2}}. \tag{3.75}$$

Again, $G_0(\xi/2) = G_0(-\xi/2)$, but $G_0(\tau)$ is not an even function.

The uniform asymptotic expansion can be obtained by using Bleistein's method [5] i.e. by substituting into (3.72)

$$G_n(\tau) = -c_n - \frac{d_n}{\tau} - \left(1 - \frac{\xi^2}{4\tau^2}\right) H_n(\tau), \tag{3.76}$$

and

$$G_{n+1}(\tau) = -\tau^{-c+2a+1} \frac{d}{d\tau} \left(\tau^{c-2a-1} H_n(\tau)\right), \tag{3.77}$$

with $n = 0$, the integral (3.72) becomes

$$\begin{aligned}
{}_2F_1\left(\begin{matrix}\lambda + a, a - \lambda\\ c\end{matrix}; \frac{1-z}{2}\right) &= L_+ \int_{\mathcal{C}_+} e^{-\lambda\left(\tau + \frac{\xi^2}{4\tau}\right)} \left(-c_0 - \frac{d_0}{\tau} - \left(1 - \frac{\xi^2}{4\tau^2}\right) H_0(\tau)\right) \tau^{c-2a-1}\, d\tau \\
&+ L_- \int_{\mathcal{C}_-} e^{-\lambda\left(\tau + \frac{\xi^2}{4\tau}\right)} \left(-c_0 - \frac{d_0}{\tau} - \left(1 - \frac{\xi^2}{4\tau^2}\right) H_0(\tau)\right) \tau^{c-2a-1}\, d\tau.
\end{aligned} \tag{3.78}$$

Now by integration by parts and using the integral representation

$$K_{\nu}(x) = \frac{1}{2}\left(\frac{x}{2}\right)^{\nu} \int_0^{\infty} e^{-t-\frac{x^2}{4t}} \frac{dt}{t^{\nu+1}}, \tag{3.79}$$

we obtain the approximation

$$\begin{aligned}
{}_2F_1\left(\begin{matrix}\lambda + a, a - \lambda\\ c\end{matrix}; \frac{1-z}{2}\right) &= 2c_0 \left(\frac{\xi}{2}\right)^{c-2a} \left(L_+ e^{(c-2a)\pi i} K_{2a-c}(-\xi\lambda) - L_- K_{2a-c}(\xi\lambda)\right) \\
&+ 2d_0 \left(\frac{\xi}{2}\right)^{c-2a-1} \left(L_+ e^{(c-2a-1)\pi i} K_{2a-c+1}(-\xi\lambda) - L_- K_{2a-c+1}(\xi\lambda)\right) \\
&+ \frac{L_+}{\lambda} \int_{\mathcal{C}_+} e^{-\lambda\left(\tau + \frac{\xi^2}{4\tau}\right)} G_1(\tau) t^{c-2a-1}\, d\tau + \frac{L_-}{\lambda} \int_{\mathcal{C}_-} e^{-\lambda\left(\tau + \frac{\xi^2}{4\tau}\right)} G_1(\tau) t^{c-2a-1}\, d\tau,
\end{aligned} \tag{3.80}$$

where the coefficients in (3.80) can be found using (3.76)

$$c_n = -\frac{G_n\left(\frac{\xi}{2}\right) + G_n\left(-\frac{\xi}{2}\right)}{2}, \qquad d_n = -\xi \frac{G_n\left(\frac{\xi}{2}\right) - G_n\left(-\frac{\xi}{2}\right)}{4}. \tag{3.81}$$

Eq. (3.80) can be further reduced to

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix}\lambda+a, a-\lambda\\ c\end{matrix}; \frac{1-z}{2}\right) &= L^* c_0 \left(\frac{\xi}{2}\right)^{c-2a} \left(e^{(c-a-\lambda)\pi i} K_{2a-c}(\xi\lambda) - e^{(\lambda-a)\pi i} K_{2a-c}(-\xi\lambda)\right) \\
&+ L^* d_0 \left(\frac{\xi}{2}\right)^{c-2a-1} \left(e^{(c-a-\lambda)\pi i} K_{2a-c-1}(\xi\lambda) - e^{(\lambda-a)\pi i} K_{2a-c-1}(-\xi\lambda)\right) \\
&+ \frac{L_+}{\lambda} \int_{\mathcal{C}_+} e^{-\lambda\left(\tau+\frac{\xi^2}{4\tau}\right)} G_1(\tau) t^{c-2a-1}\, d\tau + \frac{L_-}{\lambda} \int_{\mathcal{C}_-} e^{-\lambda\left(\tau+\frac{\xi^2}{4\tau}\right)} G_1(\tau) t^{c-2a-1}\, d\tau, \quad (3.82)
\end{aligned}
$$

where

$$
L^* = \frac{i\Gamma(c)\Gamma(\lambda+a-c+1)}{\pi\Gamma(\lambda+a)}. \quad (3.83)
$$

By using (3.75) the first two coefficients are given in (3.35).

Continuing the same process, we can obtain the result stated in Theorem (3.3.2). We omit the derivation of the $\mathcal{O}$-term as it can be obtained using the same method as in §§3.3.1.

Coefficients

To find the coefficients a_n and b_n in (3.57), we adopt the method introduced by Olde Daalhuis and Temme [30]. These coefficients can be represented in terms of Cauchy-type integrals. We first define rational functions

$$
A_0(u,\zeta) = \frac{4u}{4u^2-\zeta^2}, \qquad B_0(u,\zeta) = \frac{\zeta^2}{4u^2-\zeta^2}, \quad (3.84)
$$

and

$$
A_{n+1}(u,\zeta) = -\frac{1}{\left(1-\frac{\zeta^2}{4u^2}\right)} \left(\frac{c}{u} + \frac{d}{du}\right) A_n(u,\zeta), \quad (3.85)
$$

$$
B_{n+1}(u,\zeta) = -\frac{1}{\left(1-\frac{\zeta^2}{4u^2}\right)} \left(\frac{c}{u} + \frac{d}{du}\right) B_n(u,\zeta), \quad (3.86)
$$

for $n = 0, 1, 2, \cdots$. Since $G_0(u)$ is analytic near the saddle points $u = \pm\frac{\zeta}{2}$ we can write (3.57) as

$$
a_0 = \frac{1}{2}\frac{1}{2\pi i} \int_{\mathcal{C}} \left\{ \frac{G_0(u)}{u-\frac{\zeta}{2}} + \frac{G_0(u)}{u+\frac{\zeta}{2}} \right\} du = \frac{1}{2\pi i} \int_{\mathcal{C}} G_0(u) A_0(u,\zeta)\, du, \quad (3.87)
$$

where $\mathcal{C}$ is a contour that encircles $\pm\zeta/2$. Similarly, we have

$$b_0 = \frac{1}{2\pi i}\int_{\mathcal{C}} G_0(u)B_0(u,\zeta)\,du. \tag{3.88}$$

Corollary 3.3.1. *Let $A_0(u,\zeta), B_0(u,\zeta)$ be defined in (3.84) and $A_n(u,\zeta)$, $B_n(u,\zeta)$ be defined by the recursion in (3.85)-(3.86), then the coefficients a_n, b_n in (3.51), can be written as*

$$a_n = \frac{1}{2\pi i}\int_{\mathcal{C}} G_0(u)A_n(u,\zeta)\,du, \qquad b_n = \frac{1}{2\pi i}\int_{\mathcal{C}} G_0(u)B_n(u,\zeta)\,du, \tag{3.89}$$

where $\mathcal{C}$ is a simple closed contour which encircles the point $\tau = \pm\frac{\zeta}{2}$.

Proof. Using (3.87) we have

$$a_n = \frac{1}{2\pi i}\int_{\mathcal{C}} G_n(u)A_0(u,\zeta)\,du. \tag{3.90}$$

By substituting (3.52) and integration by parts, we obtain

$$a_n = -\frac{1}{2\pi i}\int_{\mathcal{C}} u^{-c}H_{n-1}(u)\frac{d}{du}\left(u^c A_0(u,\zeta)\right)\,du. \tag{3.91}$$

From (3.51), we have

$$a_n = -\frac{1}{2\pi i}\int_{\mathcal{C}} u^{-c}\left(\frac{1}{1-\frac{\zeta^2}{4\tau^2}}\right)\left(G_{n-1}(u) - a_{n-1} - \frac{b_{n-1}}{u}\right)\frac{d}{du}\left(u^c A_0(u,\zeta)\right)\,du. \tag{3.92}$$

Now using the fact that $A_1(u,\tau)\left(a_{n-1} + \frac{b_{n-1}}{u}\right) = \mathcal{O}(u^{-2})$ as $u \to \infty$, we obtain

$$\begin{aligned} a_n &= -\frac{1}{2\pi i}\int_{\mathcal{C}} u^{-c}\left(\frac{1}{1-\frac{\zeta^2}{4\tau^2}}\right)G_{n-1}(u)\frac{d}{du}\left(u^c A_0(u,\zeta)\right)\,du \\ a_n &= \frac{1}{2\pi i}\int_{\mathcal{C}} G_{n-1}(u)A_1(u,\zeta))\,du, \end{aligned} \tag{3.93}$$

where

$$A_1(u,\zeta) = -\frac{1}{\left(1-\frac{\zeta^2}{4u^2}\right)}\left(\frac{c}{u} + \frac{d}{du}\right)A_0(u,\zeta), \tag{3.94}$$

If we continue this process, we get the desired result i.e., the first integral representation in (3.89). ∎

These representations can be used to compute the coefficients. For example we obtain

$$A_1 = \frac{8\zeta^4}{(4u^2-\zeta^2)^3}+\frac{4(3-c)\zeta^2}{(4u^2-\zeta^2)^2}+\frac{4(1-c)}{4u^2-\zeta^2}, \qquad B_1 = \frac{8u\zeta^4}{(4u^2-\zeta^2)^3}+\frac{4u(2-c)\zeta^2}{(4u^2-\zeta^2)^2}, \quad (3.95)$$

and it follows that

$$\begin{aligned} a_1 &= \frac{\zeta}{16}\left(G_0''(\zeta/2)-G_0''(-\zeta/2)\right)+\frac{3-2c}{8}\left(G_0'(\zeta/2)+G_0'(-\zeta/2)\right)+\frac{1-2c}{\zeta^2}b_0, \\ b_1 &= \frac{\zeta^2}{32}\left(G_0''(\zeta/2)+G_0''(-\zeta/2)\right)+\frac{(3-2c)\zeta}{16}\left(G_0'(\zeta/2)-G_0'(-\zeta/2)\right). \end{aligned} \quad (3.96)$$

3.3.2 Asymptotics for large z

The third critical point is $z=\infty$. Our two main results also hold for large z but if we are only interested in finding the behavior of the function ${}_2F_1\left(\begin{matrix}\lambda+a, a-\lambda \\ c\end{matrix}; -z\right)$ as $|\lambda|\to\infty$ and $|z|$ is large, then we can use the saddle point method. An equivalent result is given in §9 of [46]. The derivation in that publication is rather complicated and the Stokes phenomenon is not really discussed.

We start by using the integral representation (15.6.2) in [28] and write

$${}_2F_1\left(\begin{matrix}a-\lambda, \lambda+a \\ c\end{matrix}; -z\right) = L\int_0^{(1+)} e^{-\lambda f(t)} g(t)\, dt, \quad (3.97)$$

where $c-a-\lambda \neq 1, 2, 3, \cdots$, $\Re(\lambda+a)>0$. The path of integration starts at $t=0$ encircles 1 once in the positive direction and returns to its starting position. The point $t=-\frac{1}{z}$ lies outside the integration contour. In (3.97) we have

$$f(t)=\ln\left(1-\frac{1}{t}\right)-\ln(1+zt), \qquad g(t)=\frac{t^{a-1}(t-1)^{c-a-1}}{(1+zt)^a}, \quad (3.98)$$

and

$$L=\frac{\Gamma(c)\Gamma(\lambda+a-c+1)}{2\pi i\Gamma(\lambda+a)}. \quad (3.99)$$

The saddle points are

$$sp_{\pm}=1\pm\zeta, \quad \text{where} \quad \zeta=\sqrt{1+\frac{1}{z}}. \quad (3.100)$$

For the phase function we choose the branch cuts between $t=0$ and $t=1$ and from

$t = -\frac{1}{z}$ to ∞. The following are the values of (3.98) at the saddle points (3.100)

$$f(sp_{\pm}) = \mp \ln\left(z\,(1+\zeta)^2\right), \tag{3.101}$$

$$g(sp_+) = z^{-a}\zeta^{c-2a-1}\,(1+\zeta)^{-1}, \qquad g(sp_-) = e^{\pm\pi i c} z^{1-a}\zeta^{c-2a-1}\,(1+\zeta), \tag{3.102}$$

according to $\Im z \gtrless 0$, and

$$f''(sp_+) = -\frac{2}{\zeta\,(1+\zeta)^2}, \qquad f''(sp_-) = \frac{2\,(z(1+\zeta))^2}{\zeta}. \tag{3.103}$$

When z is large and $|\mathrm{ph}\,\lambda| \leqslant \frac{1}{2}\pi - \delta$ then the main contribution will come from the saddle point $t = sp_+$. We combine the saddle point approximation (1.40) with $L \sim \Gamma(c)\lambda^{1-c}/(2\pi i)$ and obtain

$$ {}_2F_1\left(\begin{matrix}\lambda+a, a-\lambda\\ c\end{matrix}; -z\right) \sim \frac{\Gamma(c)}{2\sqrt{\pi}}\lambda^{1/2-c}\zeta^{c-2a-1/2}z^{-a}\left(\left(z\,(1+\zeta)^2\right)^{\lambda} + \frac{e^{\pm\pi i(c-1/2)}}{\left(z\,(1+\zeta)^2\right)^{\lambda}}\right), \tag{3.104}$$

as $|\lambda| \to \infty$ and $|\mathrm{ph}\,\lambda| \leqslant \frac{\pi}{2}$ uniformly for large $|z|$. The $\pm$ sign in the exponential term depends on the sign of $\mathrm{ph}\,\lambda$. The second term in the asymptotic approximation (3.104) is activated when the Stokes phenomenon happens in the complex z-domain. The Stokes curves can be found using

$$\Im\left(\lambda \ln\left(\zeta^2(1+\zeta)^4\right)\right) = 0. \tag{3.105}$$

After crossing the Stokes curves the second term will contribute and could be dominant when $\mathrm{ph}\,\lambda$ is close to $\pm\pi/2$.

3.4 Case 3: (0,-1,1)

3.4.1 Case (0,-1,1)

bounded z excluding -1

The asymptotic approximation for the case when z is bounded excluding $z = -1$ has been already been discussed in [25]. We copy the main details. For fixed $a, b, c \in \mathbb{C}$ and

$|\mathrm{ph}\, z| < \pi$, we have

$$ {}_2F_1\left(\begin{matrix} a, b-\lambda \\ \lambda+c \end{matrix}; -z\right) \sim \frac{2^{\lambda}\,(z+1)^{\lambda-a}\,\Gamma(\lambda+c)\Gamma(\lambda-b+1)\lambda^{\frac{1}{2}(a-1)}}{\sqrt{2\pi}z^{\lambda/2}\Gamma(2\lambda+c-b)} \times\left(U\left(a-\frac{1}{2},-\alpha\sqrt{\lambda}\right)\sum_{s=0}^{n-1}\frac{\gamma_{0,s}}{\lambda^{s}}+U\left(a-\frac{3}{2},-\alpha\sqrt{\lambda}\right)\sum_{s=0}^{n-1}\frac{\gamma_{1,s}}{\lambda^{s+1/2}}+\mathcal{O}\left(\Phi_n(\lambda,\alpha)\right)\right), \tag{3.106} $$

as $\lambda\to\infty$, in $|\mathrm{ph}\,\lambda|<\pi$, where $U(a,z)$ are parabolic cylinder functions,

$$ \frac{1}{2}\alpha^2 = -\ln\left(1-\left(\frac{z-1}{z+1}\right)^2\right), \tag{3.107} $$

such that $\Re\alpha > 0 \iff \Re\left(\frac{z-1}{z+1}\right) > 0$. The first two coefficients are

$$ \gamma_{0,0} = z^{1-c}\,(z+1)^{c-b}\left(\frac{\alpha}{z-1}\right)^{1-a}, \qquad \gamma_{1,0} = \frac{\gamma_{0,0}-2^{c-b-\frac{1}{2}}\left(\frac{\alpha(z+1)}{z-1}\right)^{a}}{\alpha}. \tag{3.108} $$

The asymptotic sequence $\{\Phi_n(\lambda,\alpha)\}$ is defined by

$$ \Phi_n(\lambda,\alpha) = \left|U\left(a-\frac{1}{2},-\alpha\sqrt{\lambda}\right)\lambda^{-n}\right| + \left|U\left(a-\frac{3}{2},-\alpha\sqrt{\lambda}\right)\lambda^{-n-\frac{1}{2}}\right|. \tag{3.109} $$

z near -1 and large z

We start with the integral representation

$$ {}_2F_1\left(\begin{matrix} a, b-\lambda \\ \lambda+c \end{matrix}; -z\right) = e^{(\lambda-b)\pi i}\frac{\Gamma(\lambda+c)\Gamma(\lambda-b+1)}{\Gamma(2\lambda+c-b)2\pi i}\int_{\infty}^{(0+)}\frac{t^{b-\lambda-1}\,(1+t)^{a-c-\lambda}}{(t+zt+1)^{a}}\,dt, \tag{3.110} $$

where $\Re(2\lambda+c-b)>0, b-\lambda\neq 1,2,3,\cdots$, and $|\mathrm{ph}\,(z+1)|<\pi$. Here the contour starts at $+\infty$ and makes a loop around 0 in the positive direction and returns to $+\infty$. Using $t=\frac{\tau-1}{2}$, we obtain

$$ {}_2F_1\left(\begin{matrix} a, b-\lambda \\ \lambda+c \end{matrix}; -z\right) = \frac{2^{2\lambda+c-b}\Gamma(\lambda+c)\Gamma(\lambda-b+1)}{(z+1)^{a}\,\Gamma(2\lambda+c-b)2\pi i}\int_{-i\infty}^{i\infty}\frac{(1-\tau)^{b-1}\,(1+\tau)^{a-c}}{(1-\tau^2)^{\lambda}\left(\tau-\frac{z-1}{z+1}\right)^{a}}\,d\tau. \tag{3.111} $$

The saddle point of the integral in (3.111) is located at $\tau = 0$ and the branch points are at $\tau = \pm 1$ and $\tau_c = \frac{z-1}{z+1}$. We choose the branch cuts in the τ-plane along $(-\infty, -1]$ and $[1, +\infty)$.

We first assume that $z \in (0,1)$ and $\operatorname{ph}\lambda \in [0, \frac{\pi}{2})$. When $\operatorname{ph}\lambda = 0$, then the steepest descent contour will be the imaginary axis, when $\operatorname{ph}\lambda \in (0, \frac{\pi}{2})$ the contour will spiral to ∞ after crossing over to the other Riemann sheets and when $\operatorname{ph}\lambda \to \frac{\pi}{2}$, then the contour collapses to the figure of eight as shown in Figure 3.4(right). Since for the moment $z \in (0,1)$, the branch point $\tau = \tau_c$ is bounded away from the steepest decent contour and thus the main contribution will come from the saddle point at $\tau = 0$. We apply the saddle point method (1.40) and obtain the asymptotic approximation i.e.

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} a, b-\lambda \\ \lambda + c \end{matrix}; -z\right) &\sim \frac{2^{2\lambda+c-b}\Gamma(\lambda+c)\Gamma(\lambda-b+1)}{(z+1)^a\,\Gamma(2\lambda+c-b)\pi i} \sum_{n=0}^{\infty} \Gamma\left(n+\tfrac{1}{2}\right) \frac{b_{2n}}{\lambda^{n+1/2}} \\
&\sim (1-z)^{-a} \sum_{n=0}^{\infty} \frac{q_n}{\lambda^n}, \qquad (3.112)
\end{aligned}
$$

as $\lambda \to \infty$ valid for $|\operatorname{ph}\lambda| < \pi/2$. The term b_0 in (3.112) is:

$$
b_0 = \frac{i}{2}\left(\frac{1-z}{z+1}\right)^{-a}, \qquad (3.113)
$$

and

$$
q_0 = 1, \qquad q_1 = \frac{az\left((b+c-a-1)(z-1)-a-1\right)}{(z-1)^2}. \qquad (3.114)
$$

The terms b_{2n} are the coefficients in the saddle point approximation (see (1.40)). The coefficients q_n are found by combining the coefficients b_n with the asymptotics of the the prefactor in front of the sum in (3.112) (combine Eq. 5.5.5 in [4] with (3.2)).

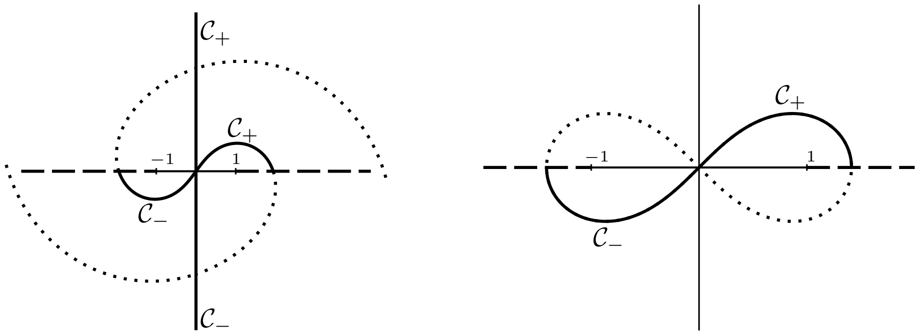

Figure 3.4: The steepest descent contours (left) when $\operatorname{ph}\lambda = 0$, the imaginary axis and a spiral when $\operatorname{ph}\lambda \in (0, \pi/2)$. The steepest descent contour (right) when $\operatorname{ph}\lambda = \pi/2$. The dotted curves show the contours on other Riemann sheets and the dashed lines show the branch cuts i.e. $(-\infty, -1]$ and $[1, \infty)$.

We will show below that the Stokes curves (see Figure 3.5) are located at

$$\Im\left(\lambda \ln\left(\frac{4z}{(z+1)^2}\right)\right) = 0. \tag{3.115}$$

When z crosses this curve then τ_c crosses the steepest descent path in the τ-plane, and a new contribution to the asymptotic approximation is born: A loop contour starting somewhere at $\tau = \infty$ and encircling $\tau = \tau_c$. The new contribution is:

$$\begin{aligned} T_\pm &= \frac{2^{2\lambda+c-b}\Gamma(\lambda+c)\Gamma(\lambda-b+1)}{(z+1)^a\,\Gamma(2\lambda+c-b)2\pi i}\int_{\pm i\infty}^{(\tau_c+)} \frac{(1-\tau)^{b-1}\,(1+\tau)^{a-c}}{(1-\tau^2)^\lambda\left(\tau-\frac{z-1}{z+1}\right)^a}\,dt \\ &= \frac{e^{\pm(a-1)\pi i}\Gamma(\lambda+c)\Gamma(\lambda-b+1)}{\Gamma(2\lambda+c-a-b+1)\Gamma(a)}\frac{(z+1)^{2\lambda+c-a-b}}{z^{\lambda+c-1}}\,{}_2F_1\left(\begin{matrix}1-a,\lambda-b+1\\2\lambda+c-a-b+1\end{matrix};z+1\right). \end{aligned} \tag{3.116}$$

Using (1.33) we have

$${}_2F_1\left(\begin{matrix}1-a,\lambda-b+1\\2\lambda+c-a-b+1\end{matrix};z+1\right) = (-z)^{a-1}\,{}_2F_1\left(\begin{matrix}1-a,\lambda+c-a\\2\lambda+c-a-b+1\end{matrix};\frac{z+1}{z}\right). \tag{3.117}$$

and thus obtain

$$T = \frac{\Gamma(\lambda+c)\Gamma(\lambda-b+1)}{\Gamma(2\lambda+c-a-b+1)\Gamma(a)}\frac{(z+1)^{2\lambda+c-a-b}}{z^{\lambda+c-a}}\,{}_2F_1\left(\begin{matrix}1-a,\lambda+c-a\\2\lambda+c-a-b+1\end{matrix};\frac{z+1}{z}\right), \tag{3.118}$$

Note that the $\pm$ in T has disappeared and that the functions in this Stokes phenomenon are connected to the linear transformation (15.10.33) in [28]

$$ {}_2F_1\left(\begin{matrix} a, b-\lambda \\ \lambda+c \end{matrix}; -z\right) = T + \frac{\Gamma(\lambda+c)\Gamma(\lambda-b+1)\,(-z)^{-a}}{\Gamma(\lambda+a-b+1)\Gamma(\lambda+c-a)} {}_2F_1\left(\begin{matrix} a, a-c-\lambda+1 \\ \lambda+a-b+1 \end{matrix}; -\frac{1}{z}\right). \tag{3.119} $$

To obtain the asymptotic approximation of the term T in (3.118), we use the following integral representation

$$ T = L\int_0^1 e^{-\lambda f(t)} g(t)\, dt, \tag{3.120} $$

with

$$ f(t) = -\ln\left(t(1-t)\right), \qquad g(t) = \frac{t^{c-a-1}\,(1-t)^{-b}}{\left(\frac{z}{z+1}-t\right)^{1-a}}, \tag{3.121} $$

and

$$ L = \frac{\Gamma(\lambda+c)}{\Gamma(\lambda+c-a)\Gamma(a)} z^{1-c-\lambda}\,(z+1)^{2\lambda+c-1-b}. \tag{3.122} $$

The saddle point is located at $t=\frac{1}{2}$. The branch points are at $t=0,1$ and $t=\frac{z}{z+1}$. When z is close to -1 the main contribution will come from the saddle point and hence by applying the saddle point method (1.40), we obtain

$$ \begin{aligned} T &\sim 2^{1-2\lambda} L \sum_{n=0}^{\infty} \Gamma\left(n+\tfrac{1}{2}\right) \frac{b_{2n}}{\lambda^{n+1/2}} \\ &\sim \frac{\sqrt{\pi}}{\Gamma(a)} 2^{b-c+1} z^{1-c}\,(z+1)^{c-a-b}\,(z-1)^{a-1}\left(\frac{(z+1)^2}{4z}\right)^{\lambda} \sum_{n=0}^{\infty} p_n \lambda^{a-n-1/2}, \end{aligned} \tag{3.123} $$

as $\lambda\to\infty$ valid for $|\mathrm{ph}\,\lambda|\leqslant\pi/2$, given that

$$ b_0 = 2^{b-c}\left(\frac{z-1}{z+1}\right)^{a-1}, \tag{3.124} $$

and

$$ p_0 = 1, \qquad p_1 = \frac{1}{4}\left((b+c)^2+b-c\right) - ab + \frac{1}{8} + \frac{a-1}{z-1}\left(2a-b-c-1+\frac{a-2}{z-1}\right). \tag{3.125} $$

The terms b_{2n} are the coefficients in a saddle point approximation (see (1.40)). The coefficients p_n are found by combining the coefficients b_n with the asymptotics (see (3.2)) of the prefactor in front of the sum in (3.123).

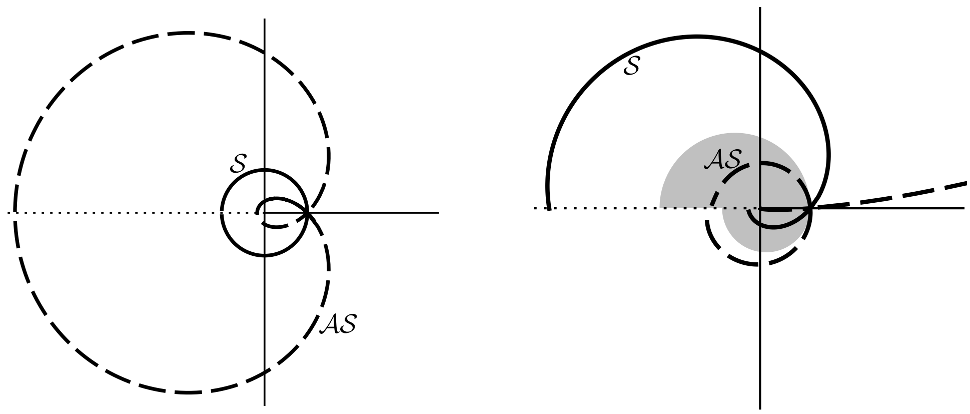

Figure 3.5: Stokes (solid lines $\mathcal{S}$) and anti-Stokes lines (dashed lines $\mathcal{AS}$) for $\mathrm{ph}\,\lambda = 0$ (left) and $\mathrm{ph}\,\lambda = e^{\frac{9}{20}\pi i}$ (right). The shaded region shows the area where the q-series in (3.126) dominates the asymptotics.

The shaded regions shown in Figure 3.5 (right) consist of two semi circles: In $\Im z > 0$, a semi-circle with radius $\frac{3}{2}$ and centre $z = -\frac{1}{2}$ and in $\Im z < 0$ a semi-circle with radius $\frac{7}{8}$ with centre $z = \frac{1}{8}$. Within these regions, the dominant contribution is from the original saddle point of the integral in (3.111) and hence we use the saddle point approximation (3.112). The contribution from the extra term T is absent until z crosses the Stokes line $\mathcal{S}$ where T is switched on and the asymptotic approximation of the exact function includes the contribution from T. On the anti-Stokes line $\mathcal{AS}$, both terms are of the same order and eventually T becomes dominant after crossing the anti-Stokes line $\mathcal{AS}$.

Thus we obtain,

$$ {}_2F_1\left(\begin{matrix} a, b-\lambda \\ \lambda + c \end{matrix}; -z\right) \sim (1-z)^{-a} \sum_{s=0}^{\infty} \frac{q_s}{\lambda^s} + \delta_g \tilde{C} \left(\frac{(z+1)^2}{4z}\right)^{\lambda} \sum_{s=0}^{\infty} \frac{p_s}{\lambda^{s+1/2-a}}, \quad (3.126) $$

as $\lambda \to \infty$, where the coefficients q_0, q_1 are given in (3.114) and p_0, p_1 are given in (3.125), and

$$ \tilde{C} = \frac{\sqrt{\pi}}{\Gamma(a)} 2^{b-c+1} (z-1)^{a-1} z^{1-c} (z+1)^{c-a-b}. \quad (3.127) $$

Since originally (say starting at $z = \frac{1}{2}$), the q-series is active, and it will switch on the

p-series when it is maximally dominant, we have to take $\delta_g = 0$ in the shaded region in Figure 3.5 and $\delta_g = 1$ to the outside of the region. This region does not depend on $\operatorname{ph}\lambda$. Note that this analysis was for $\operatorname{ph}\lambda \in [0, \frac{\pi}{2})$. For the case $\operatorname{ph}\lambda \in (-\frac{\pi}{2}, 0]$ we have to reflect the grey region in the real axis.

On the right-hand side of Figure 3.5 we illustrate the (anti)-Stokes lines in the case that $\operatorname{ph}\lambda \approx \frac{\pi}{2}$. In that figure when one travels from, say, $z = \frac{1}{2}$ to the right-hand side just below the real axis then the following happens: In (3.126) first only the q-series is active. We cross the bold-face line and via a Stokes phenomenon switch on the p-series which becomes dominant after crossing the next anti-Stokes line. A bit further an extra Stokes phenomenon happens and the p-series switches on a multiple of the q-series. This is not displayed in Figure 3.5. When $\operatorname{ph}\lambda \approx \frac{\pi}{2}$ and $z = z_0$ is close to the real axis with $\Re z_0 > 1$ and $\Im z_0 < 0$ then the factor in front of the q-series is wrong due to the extra Stokes phenomenon. Note that the line $z > 1$ is not a branch-cut for the function on the left-hand side of (3.126). Hence it should not matter how we travel from $z = \frac{1}{2}$ to $z = z_0$. We compare the results when one travels in the upper half-plane to z_0 and in that way we obtain that the factor $(1-z)^{-a}$ in front of the q-series should be replaced by $e^{a\pi i}(z-1)^{-a}$.

In general we have: when $\Re z \leqslant 1$ then (3.126) holds for $|\operatorname{ph}\lambda| \leqslant \frac{\pi}{2}$, when $\Re z > 1$ then (3.126) holds for $|\operatorname{ph}\lambda| \leqslant \frac{\pi}{2} - \delta$, and finally when $\frac{\pi}{2} - \delta \leqslant |\operatorname{ph}\lambda| \leqslant \frac{\pi}{2}$, then

$$ {}_2F_1\left(\begin{matrix} a, b-\lambda \\ \lambda + c \end{matrix}; -z\right) \sim e^{\pm a\pi i}(z-1)^{-a} \sum_{s=0}^{\infty} \frac{q_s}{\lambda^s} + \tilde{C}\left(\frac{(z+1)^2}{4z}\right)^{\lambda} \sum_{s=0}^{\infty} \frac{p_s}{\lambda^{s-a+1/2}}, \quad (3.128) $$

as $\lambda \to \infty$. The $\pm$ sign in the exponential in (3.128) corresponds to $\operatorname{ph}\lambda \gtrless 0$ and $\tilde{C}$ is defined in (3.127).

The Stokes and anti-Stokes curves in Figure 3.5 indicate that when z is large, the extra term (3.118) will always contribute since the critical point τ_c in (3.111) will coalesce with the other branch point at $\tau = 1$, so it will cross the steepest descent path which will switch on the extra term (3.118).

3.4.2 Case: (0,1,-1)

Via (15.10.25) in [28], we transform case (0,1,-1) into (0,-1,1):

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix}1-a, \lambda-b+1\\ 2-c-\lambda\end{matrix};-z\right)\\
&\quad= \frac{\Gamma(2-c-\lambda)\Gamma(2\lambda+c-b)\Gamma(a)z^{\lambda+c-1}}{\Gamma(\lambda+c)\Gamma(\lambda-b+1)\Gamma(a-c-\lambda+1)\,(z+1)^{2\lambda+c-a-b}}\,{}_2F_1\left(\begin{matrix}a, b-\lambda\\ \lambda+c\end{matrix};-z\right)\\
&\qquad+\frac{\Gamma(2\lambda+c-b)\Gamma(a)z^{\lambda+c-a-1}}{\Gamma(\lambda+c-1)\Gamma(\lambda+a-b+1)\,(z+1)^{2\lambda+c-a-b}}\,{}_2F_1\left(\begin{matrix}a, a-c-\lambda+1\\ \lambda+a-b+1\end{matrix};-\frac{1}{z}\right).
\end{aligned}
\tag{3.129}
$$

We will use this result to obtain the asymptotic approximation in the two cases discussed below.

Bounded z including 1

For z bounded, and bounded away from -1 in the sector $|\mathrm{ph}\, z| < \pi$, the uniform asymptotic approximation is given in equation (2.7) in [29] and reads

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix}1-a, \lambda-b+1\\ 2-c-\lambda\end{matrix};-z\right) &\sim \frac{2^{\lambda}z^{\lambda/2+c-1}\Gamma(2-c-\lambda)\Gamma(a)\lambda^{\frac{1}{2}(a-1)}}{\sqrt{2\pi}\,(z+1)^{\lambda+c-b}\,\Gamma(a-c-\lambda+1)}\\
&\times\left(\left\{U\left(a-\frac{1}{2},-\alpha\sqrt{\lambda}\right)-\frac{\sin((\lambda+c)\pi)}{\sin((\lambda+c-a)\pi)}U\left(a-\frac{1}{2},\alpha\sqrt{\lambda}\right)\right\}\sum_{s=0}^{\infty}\frac{\gamma_{0,s}}{\lambda^{s}}\right.\\
&\left.+\left\{U\left(a-\frac{3}{2},-\alpha\sqrt{\lambda}\right)+\frac{\sin((\lambda+c)\pi)}{\sin((\lambda+c-a)\pi)}U\left(a-\frac{3}{2},\alpha\sqrt{\lambda}\right)\right\}\sum_{s=0}^{\infty}\frac{\gamma_{1,s}}{\lambda^{s+1/2}}\right),
\end{aligned}
\tag{3.130}
$$

as $\lambda \to \infty$ in $|\mathrm{ph}\,\lambda| < \pi$ for fixed $a, b, c \in \mathbb{C}$. The first two coefficients $\gamma_{0,0}$ and $\gamma_{0,1}$ are given in (3.108) and α is given in (3.107).

For the cases that both $0 < \pm\mathrm{ph}\,(\lambda) < \pi$ and $\mp\frac{\pi}{4} \leqslant \mathrm{ph}\,(-\alpha\sqrt{\lambda}) \leqslant \mp\frac{3\pi}{4}$, the dominant asymptotic behaviour of the parabolic cylinder functions in (3.130) can be the same for certain combinations of α and λ, hence one has to following connection relations

$$\begin{aligned}
&U\left(a-\frac{1}{2},-\alpha\sqrt{\lambda}\right)-\frac{\sin((\lambda+c)\pi)}{\sin((\lambda+c-a)\pi)}U\left(a-\frac{1}{2},\alpha\sqrt{\lambda}\right)\\
&\qquad=\frac{\pm i\sqrt{2\pi}e^{\mp\pi ia/2}}{\Gamma(a)}U\left(\frac{1}{2}-a,\pm i\alpha\sqrt{\lambda}\right)-\frac{\sin(a\pi)e^{\pm(\lambda+c-a)\pi i}}{\sin((\lambda+c-a)\pi)}U\left(a-\frac{1}{2},\alpha\sqrt{\lambda}\right)\\
&U\left(a-\frac{3}{2},-\alpha\sqrt{\lambda}\right)+\frac{\sin((\lambda+c)\pi)}{\sin((\lambda+c-a)\pi)}U\left(a-\frac{3}{2},\alpha\sqrt{\lambda}\right)\\
&\qquad=\frac{-\sqrt{2\pi}e^{\mp\pi ia/2}}{\Gamma(a-1)}U\left(\frac{3}{2}-a,\pm i\alpha\sqrt{\lambda}\right)+\frac{\sin(a\pi)e^{\pm(\lambda+c-a)\pi i}}{\sin((\lambda+c-a)\pi)}U\left(a-\frac{3}{2},\alpha\sqrt{\lambda}\right).
\end{aligned}\tag{3.131}$$

z near -1 and large z

Note that the complex variable appearing in the hypergeometric functions on the right-hand side of (3.129) are $-z$ and $-1/z$. This could lead to problems while approximating these hypergeometric functions *e.g*, $0 < z < 1$ being in the upper half plane for the hypergeometric function in the first term then for the hypergeometric function in the second term z^{-1} will be located in the lower half plane such that $\Re z^{-1} > 1$ and $\Im z^{-1} < 0$ and due to the Stokes phenomenon near the real positive axis, it would be difficult to combine the results in (3.126) and (3.128) for the hypergeometric functions on the right-hand side of (3.129).

Thus to tackle this case we start with the linear transformation ((15.10.18) in [28])

$${}_2F_1\left(\begin{matrix}1-a,\lambda-b+1\\2-c-\lambda\end{matrix};-z\right)=L_1\,{}_2F_1\left(\begin{matrix}1-a,\lambda+c-a\\2\lambda+c-a-b+1\end{matrix};1+\frac{1}{z}\right)+L_2\,{}_2F_1\left(\begin{matrix}a,b-\lambda\\\lambda+c\end{matrix};-z\right),\tag{3.132}$$

where

$$L_1=(-z)^{a-1}\frac{\Gamma(\lambda+c-a)\Gamma(2\lambda+c-b)}{\Gamma(\lambda+c-1)\Gamma(2\lambda+c-a-b+1)},\tag{3.133}$$

and

$$L_2=\frac{(-z)^{\lambda+c-1}\,\Gamma(2-c-\lambda)\Gamma(\lambda+c-a)\Gamma(2\lambda+c-b)}{(z+1)^{2\lambda+c-a-b}\,\Gamma(\lambda+c)\Gamma(\lambda-b+1)\Gamma(1-a)}.\tag{3.134}$$

Writing the first term in (3.132) as

$$T_1=\frac{\Gamma(\lambda+c-a)\Gamma(2\lambda+c-b)\Gamma(a)}{\Gamma(\lambda+c-1)\Gamma(\lambda+c)\Gamma(\lambda-b+1)}\frac{(-z)^{a-1}\,z^{\lambda+c-a}}{(z+1)^{2\lambda+c-a-b}}T,\tag{3.135}$$

where T is given in (3.118), and using the result given in (3.123) we obtain

$$\begin{aligned} T_1 &\sim \frac{\Gamma(\lambda+c-a)\Gamma\left(\lambda+\frac{c-b}{2}\right)\Gamma\left(\lambda+\frac{c-b+1}{2}\right)}{\Gamma(\lambda+c-1)\Gamma(\lambda+c)\Gamma(\lambda-b+1)}(1-z)^{a-1}\sum_{s=0}^{\infty}\frac{p_s}{\lambda^{1/2-a+s}} \\ &\sim (1-z)^{a-1}\sum_{s=0}^{\infty}\frac{\tilde{p}_s}{\lambda^s}, \end{aligned} \tag{3.136}$$

as $\lambda \to \infty$ in the sector $|\mathrm{ph}\,\lambda| \leqslant \pi/2-\delta$. The first two coefficients of the p series are given in (3.125) and it follows that

$$\tilde{p}_0 = 1, \qquad \tilde{p}_1 = \frac{z(a-1)}{z-1}\left(a-b-c+1+\frac{a-2}{z-1}\right). \tag{3.137}$$

We take for the moment $0 \leqslant \mathrm{ph}\,\lambda \leqslant \frac{\pi}{2}-\delta$. Let T_2 be the second term on the right-hand side of (3.132). Using the result given in (3.126) we have

$$\begin{aligned} T_2 \sim\ & \frac{e^{\pm(\lambda+c-a-1)\pi i}\hat{C}}{\sin((\lambda+c-1)\pi)}\left(\frac{4z}{(z+1)^2}\right)^{\lambda}\sum_{s=0}^{\infty}\frac{\tilde{q}_s}{\lambda^{a-1/2+s}} \\ & +\delta_g e^{\pm(\lambda+c-1)\pi i}(z-1)^{a-1}\frac{\sin(a\pi)}{\sin((\lambda+c-1)\pi)}\sum_{s=0}^{\infty}\frac{\tilde{p}_s}{\lambda^s}, \end{aligned} \tag{3.138}$$

as $\lambda \to \infty$, where

$$\hat{C} = \frac{2^{c-b-1}\sqrt{\pi}z^{c-1}(z-1)^{-a}}{\Gamma(1-a)(z+1)^{c-a-b}}, \tag{3.139}$$

$\delta_g = 0$ inside the grey region in Figure 3.5 and $\delta_g = 1$ to the outside of that region, and $\tilde{q}_0 = 1$ and

$$\tilde{q}_1 = \frac{a(2a-c-b+2)}{1-z} - \frac{a(a+1)}{(1-z)^2} + \frac{9}{16} + \frac{b(b-2)}{2} - \frac{1}{4}\left(c+b-\frac{5}{2}\right)^2 - \frac{1}{2}\left(a-b+\frac{1}{2}\right)^2. \tag{3.140}$$

The $\pm$ sign in the exponentials in (3.138) refer to the case $\Im z \lessgtr 0$.

We continue with $0 \leqslant \mathrm{ph}\,\lambda \leqslant \frac{\pi}{2}$ and combine the asymptotic approximations given in (3.136) and (3.138) with the Stokes phenomenon discussed at the end of §3.4.1. When z

is in the grey region in Figure 3.5 and $\Im z \leqslant 0$ then we obtain

$$
\begin{aligned}
{}_2F_1\left({1-a,\lambda-b+1 \atop 2-c-\lambda};-z\right) \sim\ & (1-z)^{a-1}\sum_{s=0}^{\infty}\frac{\tilde{p}_s}{\lambda^s} \\
& +\frac{e^{\pm(\lambda+c-a-1)\pi i}\hat{C}}{\sin((\lambda+c-1)\pi)}\left(\frac{4z}{(z+1)^2}\right)^{\lambda}\sum_{s=0}^{\infty}\frac{\tilde{q}_s}{\lambda^{a-1/2+s}},
\end{aligned}
\tag{3.141}
$$

as $\lambda \to \infty$ and when z is outside that grey region

$$
\begin{aligned}
{}_2F_1\left({1-a,\lambda-b+1 \atop 2-c-\lambda};-z\right) \sim\ & \frac{\sin((\lambda+c-a)\pi)}{\sin((\lambda+c-1)\pi)}(z-1)^{a-1}\sum_{s=0}^{\infty}\frac{\tilde{p}_s}{\lambda^s} \\
& +\frac{e^{\pm(\lambda+c-a-1)\pi i}\hat{C}}{\sin((\lambda+c-1)\pi)}\left(\frac{4z}{(z+1)^2}\right)^{\lambda}\sum_{s=0}^{\infty}\frac{\tilde{q}_s}{\lambda^{a-1/2+s}},
\end{aligned}
\tag{3.142}
$$

as $\lambda \to \infty$, with a $-$ sign in the exponential term.

For the case $-\frac{\pi}{2}\leqslant \operatorname{ph}\lambda \leqslant 0$, the grey region in Figure 3.5 is reflected in the real axis, but for this region we take the same choice for the sign in the exponential in (3.141). Finally when z is outside that grey region, we use (3.142) with a $+$ sign in the exponential.

3.5 Case 4 and 5: (-1,-1,1) and (1,1,-1)

3.5.1 Case (-1,-1,1)

We will split this case in three sub cases.

bounded variable not including a neighbourhood of 1

The uniform asymptotic approximation of this case for fixed $a, b, c \in \mathbb{C}$ and $|\operatorname{ph} z| < \pi$ is in terms of Airy functions (see (3.14) in [29])

$$
\begin{aligned}
{}_2F_1\left({a-\lambda,b-\lambda \atop \lambda+c};-z\right) =\ & \frac{(z+1)^{c-a-b+3\lambda/2}}{z^{c-a+\lambda/2}}\frac{\Gamma(\lambda+c)\Gamma(\lambda-a+1)}{\Gamma(2\lambda+c-a)} \\
& \times\left(\operatorname{Ai}\left(\lambda^{2/3}x\right)\sum_{s=0}^{n-1}\frac{(-)^s\alpha_s}{\lambda^{s+\frac{1}{3}}}-\operatorname{Ai}'\left(\lambda^{2/3}x\right)\sum_{s=0}^{n-1}\frac{(-)^s\beta_s}{\lambda^{s+\frac{2}{3}}}+\mathcal{O}\left(\Phi_n(\lambda,x)\right)\right),
\end{aligned}
\tag{3.143}
$$

as $\lambda \to \infty$, $|\mathrm{ph}\,\lambda| \leqslant \frac{\pi}{2}$, where

$$\Phi_n(\lambda, x) = \left|\mathrm{Ai}\left(\lambda^{2/3}x\right)\right| |\lambda|^{-n-\frac{1}{3}} + \left|\mathrm{Ai}'\left(\lambda^{2/3}x\right)\right| |\lambda|^{-n-\frac{2}{3}}, \tag{3.144}$$

and

$$\zeta = \operatorname{arccosh}\left(\frac{1}{4z} - 1\right), \qquad \frac{4}{3}x^{3/2} = -2\zeta + 3\ln\left(\frac{2+e^{\zeta}}{2+e^{-\zeta}}\right), \tag{3.145}$$

such that $z^{-1} > 8 \iff \zeta > 0 \iff x > 0$. For the first two coefficients we have

$$\alpha_0 = \frac{G_0(\sqrt{x}) + G_0(-\sqrt{x})}{2}, \qquad \beta_0 = \frac{G_0(\sqrt{x}) - G_0(-\sqrt{x})}{2\sqrt{x}}, \tag{3.146}$$

where

$$G_0\left(\pm\sqrt{x}\right) = e^{\pm(1-a-b)\zeta/2}\left(2+e^{\pm\zeta}\right)^{a+b-c}(2z)^{(b-a+1)/2}\left(\frac{x}{1-8z}\right)^{1/4}. \tag{3.147}$$

z near 1

In this section we will assume that z and z^{-1} are bounded, and that $|\mathrm{ph}\,(1-z)| < \pi$. We start with (1.35)

$${}_2F_1\left(\begin{matrix} a-\lambda, b-\lambda \\ \lambda+c \end{matrix}; z\right) = (1-z)^{\lambda-a}\,{}_2F_1\left(\begin{matrix} a-\lambda, 2\lambda+c-b \\ \lambda+c \end{matrix}; \frac{z}{z-1}\right). \tag{3.148}$$

Using the integral representation (15.6.2) in [28] we obtain

$${}_2F_1\left(\begin{matrix} a-\lambda, b-\lambda \\ \lambda+c \end{matrix}; z\right) = \frac{\Gamma(\lambda-b+1)\Gamma(\lambda+c)}{\Gamma(2\lambda+c-b)2\pi i}\int_0^{(1+)} \frac{(t-1)^{b-\lambda-1}\,t^{2\lambda+c-b-1}}{(1-z+zt)^{a-\lambda}}\,dt, \tag{3.149}$$

where $b-\lambda \neq 1, 2, 3, \cdots$ and $\Re(2\lambda+c-b) > 0$. The path of integration starts at $t=0$ encircles 1 once in the positive direction and returns to its starting position. The point $t = 1 - z^{-1}$ lies outside the contour of integration. We write

$${}_2F_1\left(\begin{matrix} a-\lambda, b-\lambda \\ \lambda+c \end{matrix}; z\right) = L\int_0^{(1+)} e^{-\lambda f(t)}g(t)\,dt, \tag{3.150}$$

where

$$f(t) = \ln(t-1) - \ln(1-z+zt) - 2\ln(t), \qquad g(t) = \frac{(t-1)^{b-1}\,t^{c-b-1}}{(1-z+zt)^{a}}, \tag{3.151}$$

and

$$L = \frac{\Gamma(\lambda - b + 1)\Gamma(\lambda + c)}{\Gamma(2\lambda + c - b)2\pi i}. \tag{3.152}$$

The saddle points are located at

$$sp_{\pm} = \frac{4z - 1 \pm \sqrt{1+8z}}{4z}, \tag{3.153}$$

which can be presented as

$$sp_{-} = \frac{3 + \sqrt{1+8z}}{1 + \sqrt{1+8z}}, \qquad sp_{+} = \left(\frac{z-1}{z}\right)\frac{1 + \sqrt{1+8z}}{3 + \sqrt{1+8z}}. \tag{3.154}$$

The branch points of the phase function are at $t = 0$, 1 and $t = 1 - z^{-1}$. It follows from the details below that the Stokes curves are located at

$$\Im\left(\lambda \ln\left(\frac{64\,(1-z)^3\left(1+\sqrt{1+8z}\right)^2}{z\left(3+\sqrt{1+8z}\right)^6}\right)\right) = 0. \tag{3.155}$$

We note that when $\operatorname{ph}\lambda = 0$ and $0 < z < 1$, only the saddle point sp_{-} will contribute. The Stokes phenomenon will happen in the complex z-plane and switch on sp_{+}. Its contribution can be dominant when $\operatorname{ph}\lambda$ is close to $\pm\pi/2$. Thus we use the saddle point approximation (1.40) which is the combination of the asymptotic approximation at both saddle points

$$\begin{aligned}{}_2F_1\left({a-\lambda, b-\lambda \atop \lambda + c}; z\right) \sim \; & \tilde{L}\left(\left(\frac{2}{1+\sqrt{1+8z}}\right)^{\lambda+c-1}\left(\frac{3+\sqrt{1+8z}}{4}\right)^{3\lambda+c-a-b}\right. \\ & \left. + e^{\pm\pi i(\lambda+c-1/2)}\left(\frac{1+\sqrt{1+8z}}{4z}\right)^{\lambda+c-1}\left(\frac{2(1-z)}{3+\sqrt{1+8z}}\right)^{3\lambda+c-a-b}\right),\end{aligned} \tag{3.156}$$

as $\lambda \to \infty$, $|\operatorname{ph}\lambda| \leqslant \pi/2$ and $\Im z \gtrless 0$, where

$$\tilde{L} = \frac{2^{2\lambda+c-b-1}\Gamma(\lambda-b+1)\Gamma(\lambda+c)}{\Gamma(2\lambda+c-b)\,(1+8z)^{1/4}\sqrt{\lambda\pi}} \sim (1+8z)^{-1/4}, \tag{3.157}$$

as $\lambda \to \infty$.

Large z

To obtain a convenient integral representation we assume for the moment that $z > 0$ and considering the integral representation (15.6.4) in [28] that is

$$ {}_2F_1\left(\begin{matrix} a-\lambda, b-\lambda \\ \lambda+c \end{matrix}; -z\right) = e^{(\lambda-b)\pi i}\frac{\Gamma(\lambda-b+1)\Gamma(\lambda+c)}{\Gamma(2\lambda+c-b)2\pi i}\int_1^{(0+)} \frac{t^{b-\lambda-1}\,(1-t)^{2\lambda+c-b-1}}{(1+zt)^{a-\lambda}}\,dt. \tag{3.158}$$

In this integral, for the upper half plane we substitute $t = \tilde{t}e^{2\pi i}$ and take a contour $\mathcal{C}_+$ emanating from $\tilde{t} = 1$ and ending at $\tilde{t} = -\frac{1}{z}$. For the lower half plane, we consider the contour $\mathcal{C}_-$ starting from $t = -\frac{1}{z}$ to $t = 1$. Writing $t = \tilde{t}$ we have

$$ {}_2F_1\left(\begin{matrix} a-\lambda, b-\lambda \\ \lambda+c \end{matrix}; -z\right) = e^{(b-\lambda)\pi i}L\int_{\mathcal{C}_+} e^{-\lambda f(t)}g(t)\,dt + e^{(\lambda-b)\pi i}L\int_{\mathcal{C}_-} e^{-\lambda f(t)}g(t)\,dt, \tag{3.159}$$

where

$$ f(t) = -\ln\left(\frac{1}{zt}+1\right) - 2\ln(1-t), \qquad g(t) = \frac{t^{b-1}\,(1-t)^{c-b-1}}{\left(\frac{1}{z}+t\right)^a}, \tag{3.160}$$

and

$$ L = \frac{z^{\lambda-a}\Gamma(\lambda-b+1)\Gamma(\lambda+c)}{\Gamma(2\lambda+c-b)2\pi i} \sim \frac{\sqrt{\lambda/\pi}z^{\lambda-a}}{i2^{2\lambda+c-b}}, \tag{3.161}$$

as $\lambda \to \infty$. The saddle points are

$$ sp_\pm = \frac{-1 \pm i\sqrt{8z-1}}{4z}. \tag{3.162}$$

When $z \to \infty$ then saddle points will coalesce with the branch point at $t = 0$. To obtain the uniform asymptotic expansion, we use the transformation

$$ f(t) = \tau - \frac{\alpha^2}{4\tau} - \beta. \tag{3.163}$$

Corresponding to the saddle points $t = sp_\pm$ we have $\tau = \pm i\alpha/2$. Thus

$$ f(sp_\pm) = \pm i\alpha - \beta. \tag{3.164}$$

We obtain

$$ \beta = \frac{3}{2}\ln\left(\frac{z+1}{z}\right), \tag{3.165}$$

and writing $\zeta = (8z-1)^{-1/2}$,

$$\alpha = \frac{i}{2}\ln\left(\left(\frac{1-3\zeta i}{1+3\zeta i}\right)^3\left(\frac{1+\zeta i}{1-\zeta i}\right)\right). \tag{3.166}$$

Now when $z \to \infty$, then $\alpha \to 0$ and $\beta \to 0$ and thus the phase function in (3.163) reduces to $f(t) = -2\ln(1-t) = \tau$. Hence as $\tau \to 0$ we have $t \sim \frac{1}{2}\tau$ and also $g(t) = t^{b-a-1}(1-t)^{c-b-1} \sim (\tau/2)^{b-a-1}$.

Using the transformation (3.163), our integral representation in (3.159) becomes

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a-\lambda, b-\lambda \\ \lambda+c \end{matrix}; -z\right) &= e^{(b-\lambda)\pi i}L\int_{\tilde{\mathcal{C}}_+} e^{-\lambda\left(\tau-\frac{\alpha^2}{4\tau}-\beta\right)}\frac{G_0(\tau)}{\tau^{a-b+1}}\,d\tau \\ &\quad + e^{(\lambda-b)\pi i}L\int_{\tilde{\mathcal{C}}_-} e^{-\lambda\left(\tau-\frac{\alpha^2}{4\tau}-\beta\right)}\frac{G_0(\tau)}{\tau^{a-b+1}}\,d\tau, \end{aligned} \tag{3.167}$$

where the contour $\tilde{\mathcal{C}}_\pm$ is the image of $\mathcal{C}_\pm$ in the τ-plane, which starts from $\tau = 0$ and goes to ∞ in the upper and the lower half plane. Also

$$G_0(\tau) = g(t)\frac{dt}{d\tau}\tau^{a-b+1}, \tag{3.168}$$

where

$$\begin{aligned} \left(\frac{dt}{d\tau}\right)_{\tau=\pm i\alpha/2} &= \sqrt{\pm\frac{4}{\alpha f''(sp_\pm)}} \\ &= \frac{(1\mp i/\zeta)(3\pm i/\zeta)\zeta^{1/2}}{4z\sqrt{2\alpha}}, \end{aligned} \tag{3.169}$$

which can be obtained from (3.163) using l'Hôpital's rule. The values of (3.168) at the two saddle points $\tau = \pm i\alpha/2$ are

$$G_0\left(\pm\frac{i\alpha}{2}\right) = z^{1-c+a}2^{3/2-3c+a+2b}\alpha^{a-b+1/2}\zeta^{b+a-2c+3/2}(1\pm\zeta i)^{c-1}(1\mp 3\zeta i)^{c-b-a}. \tag{3.170}$$

To obtain the uniform asymptotic approximation, we use the Bleistein method [5] i.e. we substitute into (3.167)

$$G_n(\tau) = a_n + \frac{b_n}{\tau} - \left(1+\frac{\alpha^2}{4\tau^2}\right)H_n(\tau), \qquad G_{n+1}(\tau) = -\tau^{a-b+1}\frac{d}{d\tau}\left(\tau^{b-a-1}H_n(\tau)\right), \tag{3.171}$$

where

$$a_n = \frac{G_n\left(\frac{i\alpha}{2}\right) + G_n\left(-\frac{i\alpha}{2}\right)}{2}, \qquad b_n = i\alpha \frac{G_n\left(\frac{i\alpha}{2}\right) - G_n\left(-\frac{i\alpha}{2}\right)}{4}, \tag{3.172}$$

with $n = 0$, and obtain after an integration by parts

$$\begin{aligned} &{}_2F_1\left(\begin{matrix} a-\lambda, b-\lambda \\ \lambda + c \end{matrix}; -z\right) \\ &= 2Le^{\beta\lambda}\Bigg(a_0\left(\frac{\alpha}{2}\right)^{b-a}\left(e^{\pi i(a+b-2\lambda)/2}K_{a-b}(-i\alpha\lambda) - e^{\pi i(2\lambda-a-b)/2}K_{a-b}(i\alpha\lambda)\right) \\ &\quad +b_0\left(\frac{\alpha}{2}\right)^{b-a-1}\left(e^{\pi i(a+b+1-2\lambda)/2}K_{a-b+1}(-i\alpha\lambda) - e^{\pi i(2\lambda-a-b-1)/2}K_{a-b+1}(i\alpha\lambda)\right)\Bigg) \\ &\quad +\frac{L}{\lambda}\left(e^{(b-\lambda)\pi i}\int_{\tilde{\mathcal{C}}_+} e^{-\lambda\left(\tau-\frac{\alpha^2}{4\tau}-\beta\right)}\frac{G_1(\tau)}{\tau^{a-b+1}}\,d\tau + e^{(\lambda-b)\pi i}\int_{\tilde{\mathcal{C}}_-} e^{-\lambda\left(\tau-\frac{\alpha^2}{4\tau}-\beta\right)}\frac{G_1(\tau)}{\tau^{a-b+1}}\,d\tau\right). \end{aligned} \tag{3.173}$$

In the derivation of (3.173) we have used integration by parts and the integral representation given in (3.79) for the K Bessel function. Since the integrals in (3.167) and the final two integrals in (3.173) are similar, we can repeat this process and obtain a uniform asymptotic expansion. We focus only on the dominant terms, use the connection formula (10.27.8) in [33], use for L asymptotic approximation (3.161), and present our approximation in terms of Hankel functions

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a-\lambda, b-\lambda \\ \lambda + c \end{matrix}; -z\right) &\sim \frac{\sqrt{\lambda\pi}\left(1+z^{-1}\right)^{3\lambda/2} z^{\lambda-a}}{2^{2\lambda+c-b}} \\ &\times\Bigg(a_0\left(\frac{\alpha}{2}\right)^{b-a}\left(e^{\pi i(a-\lambda)}H^{(1)}_{a-b}(\alpha\lambda) + e^{\pi i(\lambda-a)}H^{(2)}_{a-b}(\alpha\lambda)\right) \\ &\quad -b_0\left(\frac{\alpha}{2}\right)^{b-a-1}\left(e^{\pi i(a-\lambda)}H^{(1)}_{a-b+1}(\alpha\lambda) + e^{\pi i(\lambda-a)}H^{(2)}_{a-b+1}(\alpha\lambda)\right)\Bigg), \end{aligned} \tag{3.174}$$

as $\lambda \to \infty$ and $|\mathrm{ph}\,\lambda| < \pi$.

To obtain the first two coefficients we combine (3.170) with (3.172) and obtain

$$\begin{aligned} a_0 &= \frac{\alpha^{a-b+1/2}\zeta^{a+b-2c+3/2}}{z^{c-a-1}2^{3c-a-2b-1/2}}\left(\frac{(1+\zeta i)^{c-1}}{(1-3\zeta i)^{a+b-c}}+\frac{(1-\zeta i)^{c-1}}{(1+3\zeta i)^{a+b-c}}\right), \\ b_0 &= \frac{i\alpha^{a-b+3/2}\zeta^{a+b-2c+3/2}}{z^{c-a-1}2^{3c-a-2b+1/2}}\left(\frac{(1+\zeta i)^{c-1}}{(1-3\zeta i)^{a+b-c}}-\frac{(1-\zeta i)^{c-1}}{(1+3\zeta i)^{a+b-c}}\right). \end{aligned} \tag{3.175}$$

3.5.2 Case (1,1-1)

Again, we split this case in three sub cases.

bounded variable not including a neighbourhood of 1

The uniform asymptotic approximation of this case for fixed $a, b, c \in \mathbb{C}$ and $|\mathrm{ph}\, z| < \pi$ is in terms of Airy functions (see (3.16) in [29])

$$\begin{aligned} {}_2F_1\left(\begin{matrix}\lambda-a+1, \lambda-b+1\\ 2-c-\lambda\end{matrix};-z\right) &= \frac{z^{a-1+\lambda/2}}{(z+1)^{3\lambda/2}}\frac{\Gamma(2-c-\lambda)\Gamma(2\lambda+c-b)}{\Gamma(\lambda-b+1)} \\ &\times\Bigg(\left(e^{(\lambda+c-4/3)\pi i}\mathrm{Ai}\left(\left(e^{\pi i}\lambda\right)^{2/3}x\right)+e^{-(\lambda+c-4/3)\pi i}\mathrm{Ai}\left(\left(e^{-\pi i}\lambda\right)^{2/3}x\right)\right)\sum_{s=0}^{n-1}\frac{(-)^s\,\alpha_s}{\lambda^{s+\frac{1}{3}}} \\ &+\left(e^{(\lambda+c-5/3)\pi i}\mathrm{Ai}'\left(\left(e^{\pi i}\lambda\right)^{2/3}x\right)+e^{-(\lambda+c-5/3)\pi i}\mathrm{Ai}'\left(\left(e^{-\pi i}\lambda\right)^{2/3}x\right)\right)\sum_{s=0}^{n-1}\frac{(-)^s\,\beta_s}{\lambda^{s+\frac{2}{3}}} \\ &+\mathcal{O}\left(\Phi_n^+(\lambda,x)+\Phi_n^-(\lambda,x)\right)\Bigg), \end{aligned} \tag{3.176}$$

as $\lambda \to \infty$, $|\mathrm{ph}\,\lambda| \leqslant \frac{\pi}{2}$, where

$$\Phi_n^{\pm}(\lambda,x) = \left|e^{\mp\lambda\pi i}\mathrm{Ai}\left(\left(e^{\pm\pi i}\lambda\right)^{2/3}x\right)\right||\lambda|^{-n-\frac{1}{3}}+\left|e^{\mp\lambda\pi i}\mathrm{Ai}'\left(\left(e^{\pm\pi i}\lambda\right)^{2/3}x\right)\right||\lambda|^{-n-\frac{2}{3}}. \tag{3.177}$$

The first two coefficients α_0, β_0 and x are given in (3.146) and in (3.145).

z near 1

We consider the integral

$${}_2F_1\left(\begin{matrix}\lambda+a, \lambda+b\\ c-\lambda\end{matrix};z\right) = \frac{\Gamma(2\lambda+b-c+1)\Gamma(c-\lambda)}{\Gamma(\lambda+b)2\pi i}\int_0^{(1+)}\frac{t^{\lambda+b-1}(t-1)^{c-b-1-2\lambda}}{(1-zt)^{\lambda+a}}\,dt. \tag{3.178}$$

Using the substitution $t-1=\tau$, we obtain

$$ {}_2F_1\left(\begin{matrix}\lambda+a,\lambda+b\\ c-\lambda\end{matrix};z\right)=\frac{\Gamma(2\lambda+b-c+1)\Gamma(c-\lambda)}{\Gamma(\lambda+b)2\pi i}\int_{-1}^{(0+)}\frac{\tau^{c-b-1-2\lambda}(\tau+1)^{\lambda+b-1}}{(1-z-z\tau)^{\lambda+a}}\,d\tau. \tag{3.179}$$

Here the path of integration starts at $\tau=-1$ encircles 0 once in the positive direction and returns to its starting position. The saddle points are located at

$$ sp_{\pm}=e^{\pm\pi i}\left(\frac{3+\sqrt{1+8z}}{1+\sqrt{1+8z}}\right),\qquad sp_0=\left(\frac{1-z}{z}\right)\left(\frac{1+\sqrt{1+8z}}{3+\sqrt{1+8z}}\right). \tag{3.180}$$

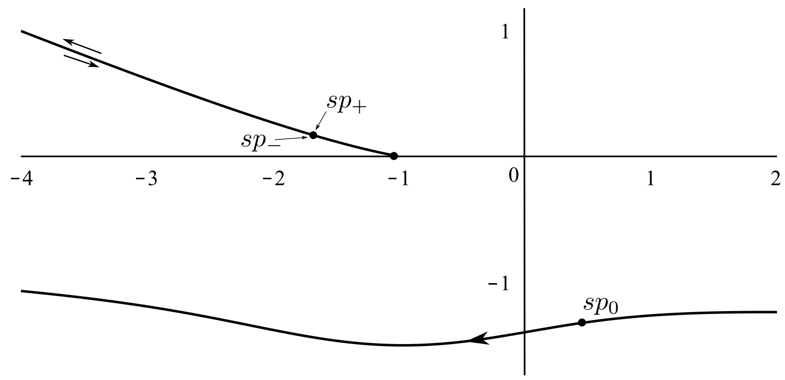

Figure 3.6: Steepest descent path when $z=0.2+0.3i$ and $\mathrm{ph}\,\lambda=0$.

The branch points of the phase function are at $\tau=0$, $\tau=-1$ and $\tau=z^{-1}-1$. The complex τ-plane has, amongst others, a branch cut from $(-\infty,0]$. The saddle point sp_+ is located on the sheet $\mathrm{ph}\,t\in(\pi/2,3\pi/2)$ and sp_- is located in $\mathrm{ph}\,t\in(-\pi/2,-3\pi/2)$.

Starting with $\mathrm{ph}\,\lambda=0$ and $z\in(0,1)$ fixed, the contour of integration can be split into three parts. Let $\mathcal{C}_+$ be the steepest descent path through sp_+, starting at -1 and finishing at ∞, let $\mathcal{C}_0$ be the steepest descent path through sp_0 starting and ending at ∞, and let $\mathcal{C}_-$ be the steepest descent path through sp_- starting at ∞ and finishing at -1. See Figure (3.6).

Hence, all these saddle points contribute, and via the saddle point method we obtain

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix}\lambda+a,\lambda+b\\ c-\lambda\end{matrix};z\right) \sim L^*\Bigg(&\left(\frac{3+\sqrt{1+8z}}{2(1-z)}\right)^{3\lambda+a+b-c}\left(\frac{1+\sqrt{1+8z}}{z}\right)^{c-\lambda-1} && \text{(3.181a)}\\
&+e^{\pi i(c-\lambda-1/2)}\left(\frac{3+\sqrt{1+8z}}{4}\right)^{c-b-a-3\lambda}\left(\frac{1+\sqrt{1+8z}}{8}\right)^{\lambda-c+1} && \text{(3.181b)}\\
&+e^{-\pi i(c-\lambda-1/2)}\left(\frac{3+\sqrt{1+8z}}{4}\right)^{c-b-a-3\lambda}\left(\frac{1+\sqrt{1+8z}}{8}\right)^{\lambda-c+1}\Bigg), && \text{(3.181c)}
\end{aligned}
$$

where

$$L^* = \frac{2^{1-b-c}\Gamma(c-\lambda)\Gamma(2\lambda+b-c+1)}{\Gamma(\lambda+b)\,(1+8z)^{1/4}\sqrt{\lambda\pi}}. \tag{3.182}$$

The asymptotic approximation can be simplified as follows

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix}\lambda+a,\lambda+b\\ c-\lambda\end{matrix};z\right) \sim L^*\Bigg(&\left(\frac{3+\sqrt{1+8z}}{2(1-z)}\right)^{a+b-c+3\lambda}\left(\frac{1+\sqrt{1+8z}}{z}\right)^{c-\lambda-1}\\
&+2\cos(\pi(\lambda-c+\tfrac{1}{2}))\left(\frac{3+\sqrt{1+8z}}{4}\right)^{c-b-a-3\lambda}\left(\frac{1+\sqrt{1+8z}}{8}\right)^{1-c+\lambda}\Bigg),
\end{aligned}
\tag{3.183}
$$

as $\lambda \to \infty$ and $|\mathrm{ph}\,\lambda| \leqslant \pi/2$.

It follows that a Stokes phenomenon will take place when

$$\Im\left(\lambda\left(\frac{\left(3+\sqrt{1+8z}\right)^6 z}{(z-1)^3\left(1+\sqrt{1+8z}\right)^2}\right)\right) = 0. \tag{3.184}$$

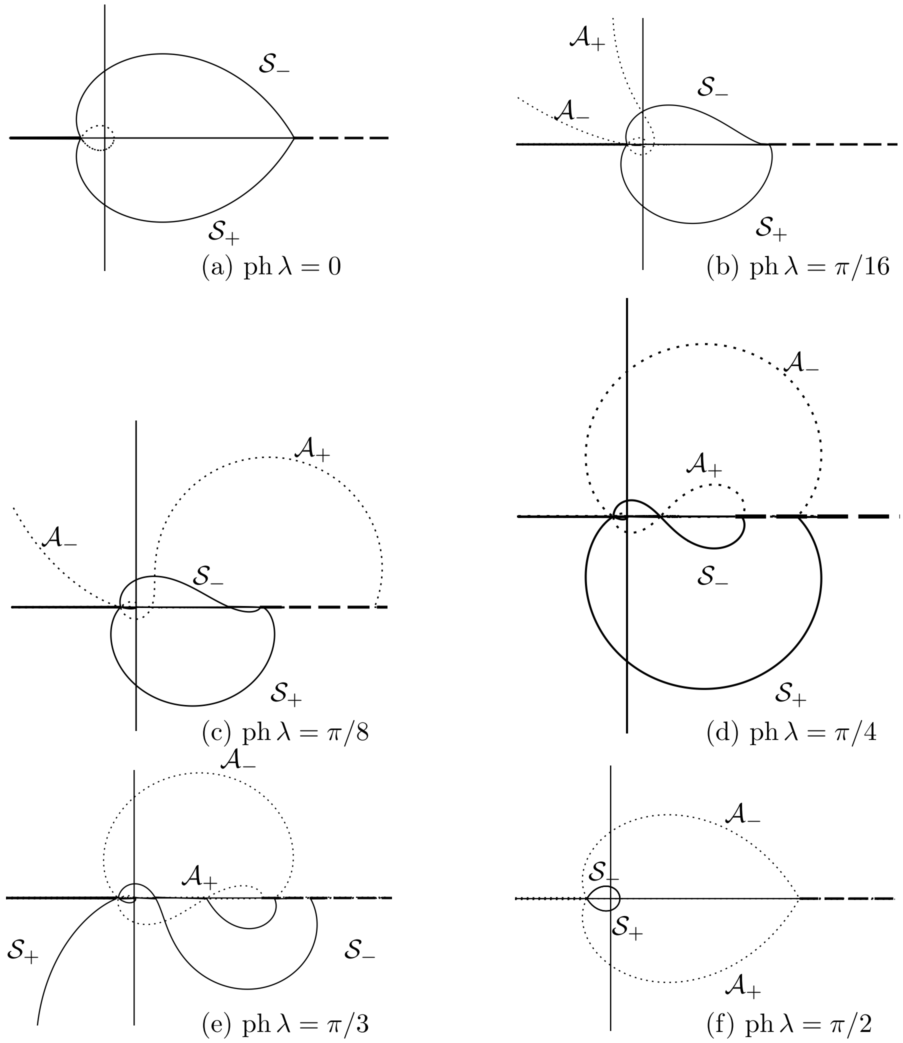

Figure 3.7: The solid lines are the Stokes lines and the dotted lines represents the anti-Stokes lines in the complex z-plane.

Figure 3.7 represents the Stokes and the anti-Stokes Curves. In the regions bounded by $\mathcal{S}_+, \mathcal{S}_-$ and possibly the branch-cut $[1, \infty)$, all three saddle points contribute. $\mathcal{S}_\pm$ are the Stokes curves where sp_0 switches off the contribution of $sp_\pm$. The anti-Stokes curves $\mathcal{A}_\pm$ are the curves where the contribution of $sp_\pm$ start dominating the contribution of sp_0. Thus in Figure 3.7b, in the second quadrant in the region bounded by $\mathcal{A}_+$, $\mathcal{S}_-$ and $\operatorname{ph} z = \pi$, contribution from (3.181b) dominates (3.181a) and (3.181c) is not active.

Large z

For the moment we take $z > 0$, $\operatorname{ph}\lambda = 0$ and replace in the integral representation (3.179) z by $-z$. We take for the contour of integration $\tau = -1 + ir$, $r \in (-\infty, \infty)$. For the contour in the upper/lower half plane, we use the substitution $r = e^{\pm\pi i}t$ and obtain

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a+\lambda, b+\lambda \\ c-\lambda \end{matrix}; -z\right) &= \frac{z^{-a-\lambda}\Gamma(1+b-c+2\lambda)\Gamma(c-\lambda)}{\Gamma(b+\lambda)2\pi i} \\ &\times \left(e^{(c-b-1-2\lambda)\pi i} \int_1^{-i\infty} \frac{t^{c-b-1-2\lambda}(1-t)^{b-1+\lambda}}{(1+z^{-1}-t)^{a+\lambda}}\, dt \right. \\ &\left. -e^{-(c-b-1-2\lambda)\pi i} \int_1^{i\infty} \frac{t^{c-b-1-2\lambda}(1-t)^{b-1+\lambda}}{(1+z^{-1}-t)^{a+\lambda}}\, dt \right). \end{aligned} \tag{3.185}$$

We write

$$ {}_2F_1\left(\begin{matrix} a+\lambda, b+\lambda \\ c-\lambda \end{matrix}; -z\right) = e^{(c-b-1-2\lambda)\pi i} L \int_{\mathcal{C}_+} e^{-\lambda f(t)} g(t)\, dt + e^{(b-c+1+2\lambda)\pi i} L \int_{\mathcal{C}_-} e^{-\lambda f(t)} g(t)\, dt, \tag{3.186}$$

where

$$ f(t) = \ln\left(\frac{z+1}{z} - t\right) - \ln(1-t) + 2\ln(t), \qquad g(t) = \frac{t^{c-b-1}(1-t)^{b-1}}{(1+z^{-1}-t)^{a}}, \tag{3.187}$$

and

$$ L = \frac{z^{-a-\lambda}\Gamma(1+b-c+2\lambda)\Gamma(c-\lambda)}{\Gamma(b+\lambda)2\pi i}. \tag{3.188}$$

The branch points of the phase function are located at $t = 0$, $t = 1$ and $t = 1 + z^{-1}$. The saddle points are located at

$$ sp_{\pm} = \frac{1 + 4z \pm i\sqrt{8z-1}}{4z}. \tag{3.189}$$

When $z \to \infty$, then both saddle points will coalesce with the branch point at $t = 1$. Hence to obtain the uniform asymptotic expansion, we use the transformation

$$ f(t) = \tau - \frac{\alpha^2}{4\tau} + \beta. \tag{3.190}$$

Corresponding to the saddle points $t = sp_{\pm}$ in (3.189) we have $\tau = \pm\frac{i\alpha}{2}$. Thus

$$f(sp_{\pm}) = \pm i\alpha + \beta. \tag{3.191}$$

We obtain that α and β are the same as in (3.166) and (3.165), where again $\zeta = (8z-1)^{-1/2}$.

Now when $z \to \infty$ then $\alpha \to 0$ and $\beta \to 0$, thus the phase function in (3.190) reduces to $f(t) = 2\ln(t) = \tau$. Hence, as $\tau \to 0$ we have $t \sim 1$, $\frac{dt}{d\tau} \sim \frac{1}{2}$, and also $g(t) = t^{c-b-1}(1-t)^{b-a-1} \sim (-\tau/2)^{b-a-1}$. Using the transformation (3.190), our integral representation in (3.186) becomes

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a+\lambda, b+\lambda \\ c-\lambda \end{matrix}; -z\right) &= e^{(c-b-1-2\lambda)\pi i} L \int_{\tilde{\mathcal{C}}_+} e^{-\lambda\left(\tau - \frac{\alpha^2}{4\tau} + \beta\right)} G_0(\tau)\,(-\tau)^{b-a-1}\, d\tau \\ &\quad - e^{(b-c+1+2\lambda)\pi i} L \int_{\tilde{\mathcal{C}}_-} e^{-\lambda\left(\tau - \frac{\alpha^2}{4\tau} + \beta\right)} G_0(\tau)\,(-\tau)^{b-a-1}\, d\tau, \end{aligned} \tag{3.192}$$

where $\tilde{\mathcal{C}}_+$ is the contour emanating from 0 and going to $\tau = \infty$ in the upper half plane and the contour $\tilde{\mathcal{C}}_-$ starts from $\tau = 0$ to $\tau = \infty$ in the lower half plane. Also we have

$$G_0(\tau) = g(t)\frac{dt}{d\tau}(-\tau)^{a-b+1}, \tag{3.193}$$

where

$$\left(\frac{dt}{d\tau}\right)_{\tau=\pm i\alpha/2} = \sqrt{\pm\frac{4}{\alpha f''(sp_{\pm})}} = \frac{(1\mp i\zeta)(1\pm 3i\zeta)}{2^{5/2}\zeta^{3/2} z\alpha^{1/2}}, \tag{3.194}$$

which can be obtained from (3.190) using l'Hôpital's rule. The following values are needed to to compute the values of (3.193) at the two saddle points.

$$g(sp_{\pm}) = e^{\mp\pi i(a-b+1)/2} 2^{5+2a-3c+b} z^{2+a-c} \zeta^{3+b+a-2c} (1\mp i\zeta)^{c-2} (1\pm 3i\zeta)^{c-b-1-a}, \tag{3.195}$$

and

$$f''(sp_{\pm}) = \mp\frac{128 i\zeta^3 z^2}{(1\mp i\zeta)^2 (1\pm 3i\zeta)^2}. \tag{3.196}$$

Combining (3.195), (3.196) and (3.194), we have

$$G_0\left(\pm i\frac{\alpha}{2}\right) = 2^{3/2+a-3c+2b} z^{1+a-c} \zeta^{3/2+b+a-2c} \left(1 \mp i\zeta\right)^{c-1} \left(1 \pm 3i\zeta\right)^{c-b-a} \alpha^{a-b+1/2}. \tag{3.197}$$

To obtain the uniform asymptotic approximation, we use Bleistein's method [5] i.e. we substitute into (3.192)

$$G_n(\tau) = c_n + \frac{d_n}{\tau} - \left(1 + \frac{\alpha^2}{4\tau^2}\right) H_n(\tau), \qquad G_{n+1}(\tau) = -\tau^{a-b+1} \frac{d}{d\tau}\left(\tau^{b-a-1} H_n(\tau)\right). \tag{3.198}$$

We skip the details since they are very similar to the details in §3.5.1 and give just the main approximation

$$\begin{aligned}
{}_2F_1\left(\begin{matrix} a+\lambda, b+\lambda \\ c-\lambda \end{matrix}; -z\right) \sim \\
\frac{L\pi i}{\left(1+z^{-1}\right)^{3\lambda/2}} & \left(c_0 \left(\frac{\alpha}{2}\right)^{b-a} \left(e^{(c-b-2\lambda)\pi i} H^{(1)}_{a-b}(\alpha\lambda) + e^{(b-c+2\lambda)\pi i} H^{(2)}_{a-b}(\alpha\lambda)\right)\right. \\
& \left. -d_0 \pi i \left(\frac{\alpha}{2}\right)^{b-a-1} \left(e^{(c-b-2\lambda)\pi i} H^{(1)}_{a-b+1}(\alpha\lambda) + e^{(b-c+2\lambda)\pi i} L H^{(2)}_{a-b+1}(\alpha\lambda)\right)\right),
\end{aligned} \tag{3.199}$$

as $\lambda \to \infty$ and $|\mathrm{ph}\,\lambda| < \pi$, where $c_0 = a_0$ and $d_0 = -b_0$, with a_0 and b_0 given in (3.175).

Chapter 4

Uniform asymptotic approximations for the Meixner-Sobolev polynomials

4.1 Introduction

The monic Meixner-Sobolev polynomials $S_n(x)$ are orthogonal with respect to the discrete inner product

$$(p, q)_S = \sum_{k=0}^{\infty} \{p(k)q(k) + \lambda \Delta p(k) \Delta q(k)\} \frac{c^k (\beta)_k}{k!}, \tag{4.1}$$

where $0 < c < 1$, $\beta > 0$, $\lambda \geqslant 0$, and Δ is the usual forward difference operator defined by $\Delta p(k) = p(k+1) - p(k)$. When $\lambda = 0$ the Meixner-Sobolev polynomials reduce to the classical Meixner polynomials. These polynomials were introduced in [2], and a recurrence relation involving S_n, S_{n-1} and 2 classical Meixner polynomials is given in [3] and [24]. This recurrence relation is very useful for generating the polynomials. The large n asymptotics in [3] is for non-oscillatory regions.

The following are the first three polynomials

$$S_0(x) = 1, \tag{4.2}$$

$$S_1(x) = \frac{\beta c + cx - x}{c}, \tag{4.3}$$

$$S_2(x) = \frac{1}{2}\left(\beta(\beta+1) + \frac{\lambda\,(c-1)^2\,(\beta-1)}{c} + x\left(\frac{(c-1)(2\beta c+c+1)}{c^2} + \frac{\lambda\,(c-1)^3\,(2\beta c-c+1)}{c^3\beta}\right) + x^2\left(\frac{(c-1)^2}{c^2} + \frac{\lambda\,(c-1)^4}{c^3\beta}\right)\right). \tag{4.4}$$

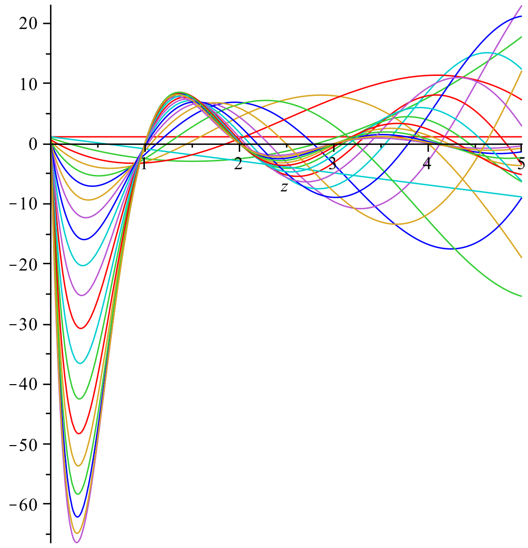

Figure 4.1: The re-scaled graph of the first few Meixner-Sobolev polynomials $\frac{S_n(z)}{\Gamma((n+3)/2)}$

In this chapter, we give large n asymptotic approximations that are valid on the positive real x axis, hence, they include the oscillatory region $0 \leqslant x/n < \frac{1+\sqrt{c}}{1-\sqrt{c}}$. We don't consider the negative real x axis since there are no zeros when $x < 0$ and hence $S_n(x) \to \infty$ as $x \to -\infty$.

The starting point is the generating function given in [24]:

$$G(t) = \sum_{n=0}^{\infty} S_n(x) t^n, \tag{4.5}$$

where

$$G(t) = \frac{1}{1-t} \frac{\left(1-\frac{t}{a}\right)^{-x-\beta+\gamma+1}}{\left(1-\frac{t}{ac}\right)^{-x}(1-act)^{\gamma}} {}_2F_1\left(\begin{matrix}-x,\gamma\\ \beta-1\end{matrix};z(t)\right), \tag{4.6}$$

in which

$$z(t) = \frac{-t(1-c)(1-a^2c)}{(1-act)(ac-t)}, \qquad \Longrightarrow \qquad 1-z(t) = \frac{(1-at)(1-\frac{t}{a})}{(1-act)(1-\frac{t}{ac})}, \tag{4.7}$$

and

$$\gamma = \frac{(1-a)(\beta-1)}{1-a^2c}, \qquad a = \frac{1+\eta c-\sqrt{(1+\eta c)^2-4c}}{2c}, \tag{4.8}$$

where $\eta = 1+\lambda\left(1-\frac{1}{c}\right)^2 > 1$. Note that $0 < a < 1$ and that $ca^2-(1+\eta c)a+1 = 0$. In (4.6) the function ${}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};t\right)$ is the standard Gauss hypergeometric function. (See [28].) Note that the definition of γ differs from the one used in [24].

From the generating function $G(t)$ we will obtain an integral representation for $S_n(x)$. The integrand will involve the Gauss hypergeometric function, and it seems not easy to obtain asymptotic approximations. However, we will use some of the linear transformations for hypergeometric functions, to express $S_n(x)$ as a sum of 3 integrals in which the hypergeometric functions can be approximated uniformly by unity. Hence, the result is a simple integral approximation where standard methods can be used.

The structure of the chapter is as follows. In §4.2, we study the asymptotics as $n\to\infty$ and x is bounded. The analysis is based on Darboux's method. We will obtain simple asymptotic approximations, and observe that the small zeros of $S_n(x)$ are located at $x = 0,1,2,\cdots$, with an exponentially small error.

In §4.3 we use several transformations to obtain an integral representation that can be used for the uniform asymptotic approximations. We discuss the location of the saddle points of the phase functions of the integrals. When $x/n = Y_\pm$, where

$$Y_- = \frac{1-\sqrt{c}}{1+\sqrt{c}}, \qquad \text{and} \qquad Y_+ = \frac{1+\sqrt{c}}{1-\sqrt{c}}, \tag{4.9}$$

these saddle points will coalesce, and Airy functions are needed in the uniform asymptotic approximations.

The multi-valuedness of the hypergeometric functions is discussed in §4.4, and we conclude that along the contours of integration of the integrals given in §4.3 the hypergeometric functions can be approximated uniformly by unity.

In the following 4 sections we discuss, the uniform asymptotic approximations. Only one saddle point dominates in the case $Y_- < x/n < Y_+$ and a simple saddle point method approximation is given in §4.5, where we also give a relative simple formula for the location of the zeros.

The cases $0 \leqslant x/n < Y_-$, $0 < x/n < Y_+$ and $Y_- < x/n$ are discussed in the next 3 sections. In the first case we give a uniform asymptotic approximation in terms of a gamma function, and in the other 2 cases Airy functions are needed in the asymptotic approximations. The later case also gives information about the location of the large zeros of $S_n(x)$, which are discussed in the final section. Our three term asymptotic approximation for the large zeros is in terms of the zeros of the Airy function $\mathrm{Ai}(z)$. When we let $\lambda \to 0$, that is, $a \to 1$, we obtain a three term asymptotic approximation for the large zeros of the classical Meixner polynomials, and our result agrees with [16], in which a two term asymptotic approximation is given. The additional term in our approximation is surprisingly simple.

In Figure 4.2 we display the graph of

$$\frac{\frac{2}{3} n S_n(x)}{\dfrac{\operatorname{erfc}(20(Y_- - y))\Gamma(x+1)}{\sqrt{n} a^n c^{(n+x)/2}\Gamma(x+\gamma+1)} + \dfrac{\operatorname{erfc}(20(y - Y_-))\Gamma(x+1)\Gamma(n-x-\gamma)}{n!(ac)^n(1-c)^x}}. \tag{4.10}$$

With this rescaling the function is $O(1)$ on the whole interval. It clearly displays the oscillatory region $x \in [0, nY_+]$, and the dramatic change near $x \approx nY_-$. Note that the graph does not include $x < 0$, since it is known that there are no zeroes on the negative real axis.

4.2 Large n and fixed x asymptotics

For the large n asymptotics we will study the singularities of $G(t)$ in the complex t plane. These are at

$$ac, \quad a, \quad 1, \quad 1/a, \quad 1/ac, \quad \text{with } 0 < ac < a < 1 < 1/a < 1/ac. \tag{4.11}$$

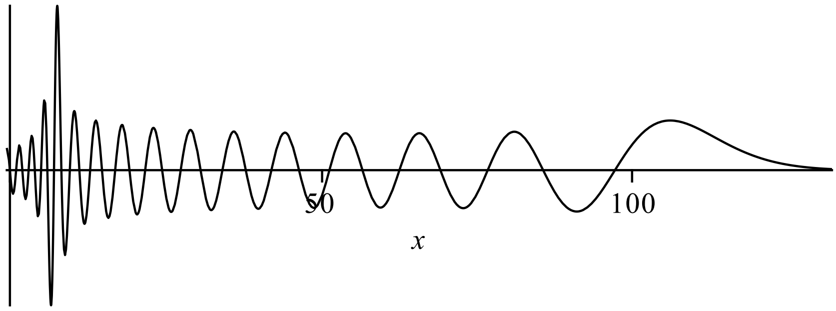

Figure 4.2: The graph of a rescaled version of $S_{30}(x)$, where $a = \frac{5}{6}$, $c = \frac{1}{3}$ and $\beta = \frac{9}{8}$. Note the dramatic changes at $x = nY_- \approx 8$ and at $x = nY_+ \approx 112$.

Hence, the singularity that is nearest to the origin is $t = ac$. For an asymptotic expansion that holds as $n \to \infty$ and a, c, β and x fixed, we need the local expansion of $G(t)$ at this singularity. It is convenient to start with the identity

$$G(t) = G_1(t) + G_2(t), \tag{4.12}$$

(use Eq. 15.8.2 in [28]), where

$$G_1(t) = K_1 \frac{t^{\gamma-\beta+1}(1-at)^{x+\beta-\gamma-1}}{(1-t)\,(1-act)^{x+\gamma}}\, {}_2F_1\left(\begin{matrix} 1-\gamma, \beta-\gamma-1 \\ 1-\gamma-x \end{matrix}; \frac{1}{z(t)}\right), \tag{4.13}$$

with

$$K_1 = \left((1-c)(\frac{1}{ac}-a)\right)^{\gamma-\beta+1} \frac{\Gamma(\beta-1)\Gamma(x+\gamma)}{\Gamma(\gamma)\Gamma(x+\beta-1)}, \tag{4.14}$$

and

$$G_2(t) = K_2 \frac{t^{-\gamma}\left(1-\frac{t}{a}\right)^{-x-\beta+\gamma+1}}{(1-t)\left(1-\frac{t}{ac}\right)^{-x-\gamma}}\, {}_2F_1\left(\begin{matrix} \gamma, 2+\gamma-\beta \\ 1+\gamma+x \end{matrix}; \frac{1}{z(t)}\right), \tag{4.15}$$

with

$$K_2 = \left((1-c)(\frac{1}{ac}-a)\right)^{-\gamma} \frac{\Gamma(\beta-1)\Gamma(-x-\gamma)}{\Gamma(\beta-\gamma-1)\Gamma(-x)}. \tag{4.16}$$

The function $G_1(t)$ has no singularity at $t = ac$. Hence, the main contributions to the large n asymptotics will come from $G_2(t)$. We expand

$$G_2(t) = K_2 \sum_{m=0}^{\infty} b_m \left(1 - \frac{t}{ac}\right)^{m+x+\gamma}, \tag{4.17}$$

with

$$b_0 = \frac{(1-c)^{-x-\beta+\gamma+1}\,(ac)^{-\gamma}}{1-ac}, \tag{4.18}$$

and obtain via Darboux's method (see §2.10(iv) in [34])

$$\begin{aligned} S_n(x) &\sim K_2 \sum_{m=0}^{\infty} \frac{b_m}{2\pi i} \oint_{\{0\}} \frac{\left(1-\frac{t}{ac}\right)^{m+x+\gamma}}{t^{n+1}}\,dt \\ &\sim K_2 \frac{(ac)^{-n}}{n!} \sum_{m=0}^{\infty} b_m \,(-m-\gamma-x)_n\,, \end{aligned} \tag{4.19}$$

as $n \to \infty$, where the Pochhammer symbol is $(z)_n = z(z+1)(z+2)\cdots(z+n-1)$.

One can observe that since

$$\frac{(-m-\gamma-x)_{n+1}}{(-m-\gamma-x)_n} = \frac{m+1+\gamma+x}{m+1+\gamma+x-n} = \mathcal{O}(1/n), \tag{4.20}$$

it follows that this series (4.19) has an asymptotic property.

Just to illustrate: Taking $a=\frac{5}{6}$, $c=\frac{1}{3}$ and $\beta=\frac{9}{8}$, then $S_{50}(1.5) = 3.256665\times 10^{23}$ and taking 4 terms on the right-hand side of (4.19) gives us the approximation 3.256592×10^{23}. Hence, the relative error is 0.000023.

It follows from the gamma function $\Gamma(-x)$ in (4.16) that $S_n(x)$ has zeros at approximately $x=m$, where m is a bounded nonnegative integer. In fact, since K_2 multiplies all terms in (4.19) and the contribution of $G_1(t)$ to the asymptotics of $S_n(x)$ will be exponentially small, it follows from Theorem 1 in [15] that the small zeros are located approximately at $x=m$ with an exponentially small error. In [16] the authors make the same observation for the small zeros of the classical Meixner polynomials.

4.3 Large n and x

Our analysis will be based on the observation that (see §15.12(ii) in [28])

$${}_2F_1\left({A, B \atop x+C}; z\right) = 1 + O(1/x), \qquad x \to \infty, \quad |\mathrm{ph}(1-z)| < \pi. \tag{4.21}$$

We will use this result for $x \to +\infty$. The result even holds for $x \to \infty$ in the sector $|\mathrm{ph} x| \leqslant \frac{1}{2}\pi - \varepsilon$, (where ε is an arbitrary small positive constant), with the same restriction on z. For larger sectors in the complex x-plane the z-region of validity will be smaller.

The hypergeometric function in the right-hand side of (4.15) is already of the form (4.21). For the one in the representation (4.13) of $G_1(t)$ we need one more transformation.

Note that the function $G_1(t)$ is analytic at $t = ac$. For that function we push t to $ac < \Re t < a$, and we use the transformation

$$G_1(t) = G_3(t) - G_4(t), \tag{4.22}$$

(combine (15.8.4) with (15.8.1) in [28]), where

$$G_3(t) = K_3 \frac{t^{\gamma-\beta+1}(1-at)^{x+\beta-\gamma-1}}{(1-t)\,(1-act)^{x+\gamma}} {}_2F_1\left({\beta-\gamma-1, 1-\gamma \atop x+\beta-\gamma}; 1-\frac{1}{z(t)}\right) \tag{4.23}$$

with

$$K_3 = \left((1-c)\left(\frac{1}{ac}-a\right)\right)^{\gamma-\beta+1} \frac{\Gamma(\beta-1)\Gamma(x+1)}{\Gamma(\gamma)\Gamma(x+\beta-\gamma)}, \tag{4.24}$$

and

$$G_4(t) = K_4 \frac{t^{-\gamma}\left(1-\frac{t}{a}\right)^{-x-\beta+\gamma+1}}{(1-t)\left(\frac{t}{ac}-1\right)^{-x-\gamma}} {}_2F_1\left({\gamma, 2+\gamma-\beta \atop 1+\gamma+x}; \frac{1}{z(t)}\right), \tag{4.25}$$

with

$$K_4 = \left((1-c)\left(\frac{1}{ac}-a\right)\right)^{-\gamma} \frac{\Gamma(\beta-1)\Gamma(x+1)\Gamma(-x-\gamma)}{\Gamma(\gamma)\Gamma(1-\gamma)\Gamma(\beta-\gamma-1)}. \tag{4.26}$$

We observe that $G_2(t)$ and $G_4(t)$ are in terms of the same hypergeometric function, but that in $G_2(t)$ we start at $0 < t < ac$ and in $G_4(t)$ we consider $ac < t < a$. Combining these two functions we obtain

$$G_{\pm}(t) = G_2(t) - G_4(t) = e^{\mp x\pi i} K \left(\frac{\frac{t}{ac}-1}{1-\frac{t}{a}}\right)^{x} g(t) {}_2F_1\left({\gamma, 2+\gamma-\beta \atop 1+\gamma+x}; \frac{1}{z(t)}\right), \tag{4.27}$$

valid in the half-plane $\pm\Im(t) > 0$, where

$$K = \left((1-c)\left(\frac{1}{ac}-a\right)\right)^{-\gamma} \frac{\Gamma(\beta-1)\Gamma(x+1)}{\Gamma(\beta-\gamma-1)\Gamma(x+1+\gamma)}, \tag{4.28}$$

and

$$g(t) = \frac{t^{-\gamma}\left(1-\frac{t}{a}\right)^{\gamma-\beta+1}}{(1-t)\left(\frac{t}{ac}-1\right)^{-\gamma}}. \tag{4.29}$$

Combining these results we obtain

$$\begin{aligned} S_n(x) &= \frac{1}{2\pi i}\oint_{\{0\}} \frac{G(t)}{t^{n+1}}\,dt \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}_3} \frac{G_3(t)}{t^{n+1}}\,dt + \frac{1}{2\pi i}\int_{\mathcal{C}_+} \frac{G_+(t)}{t^{n+1}}\,dt - \frac{1}{2\pi i}\int_{\mathcal{C}_-} \frac{G_-(t)}{t^{n+1}}\,dt \end{aligned} \tag{4.30}$$

where $\mathcal{C}_3$ is for the moment a vertical contour that crosses the real t-axis in the interval (ac, a) and $\mathcal{C}_\pm$ is a contour that emanates from $t = ac$ and goes to infinity in the upper/lower half-plane. Again, the integrands in (4.30) have singularities at the points mentioned in (4.11), and possibly a branch-point at $t = 0$.

From here onwards we will take

$$x = ny, \quad \text{where } \; y > 0, \;\; \text{bounded.} \tag{4.31}$$

Hence, we can now replace the hypergeometric functions by unity and observe that the phase-function for integrand $G_3(t)$ is $f_3(t)$ and for integrands $G_\pm(t)$ it is $f(t)$ where

$$f_3(t) = \ln(t) + y\ln\left(\frac{1-act}{1-at}\right), \quad f(t) = \ln(t) + y\ln\left(\frac{1-\frac{t}{a}}{\frac{t}{ac}-1}\right). \tag{4.32}$$

Hence, the saddle points are located at

$$\begin{aligned} Sp_{3\pm} &= \frac{1}{2ac}\left(y(c-1)+c+1 \pm \sqrt{(y(c-1)+c+1)^2-4c}\right), \\ Sp_{\pm} &= \frac{a}{2}\left(y(c-1)+c+1 \pm \sqrt{(y(c-1)+c+1)^2-4c}\right). \end{aligned} \tag{4.33}$$

Note that the sadde points 'coalesce' when $y^2 + 2y\frac{c+1}{c-1} + 1 = 0$, that is, when $y = Y_\pm$, where $Y_\pm$ are defined in (4.9).

One very useful observation: let

$$\cos\theta = \frac{1+c-(1-c)y}{2\sqrt{c}}, \qquad \text{then} \qquad Sp_\pm = a\sqrt{c}e^{\pm\theta i}, \qquad Sp_{3\pm} = \frac{1}{a\sqrt{c}}e^{\pm\theta i}. \tag{4.34}$$

It follows from these representations that for $0 < y < Y_-$ we have

$$ac < Sp_+ < a\sqrt{c} < Sp_- < a < 1 < \frac{1}{a} < Sp_{3+} < \frac{1}{a\sqrt{c}} < Sp_{3-} < \frac{1}{ac},$$

(compare (4.43)), for $Y_- \leqslant y \leqslant Y_+$ we have $|Sp_\pm| = a\sqrt{c} < 1 < |Sp_{3\pm}| = 1/(a\sqrt{c})$, and finally for $y > Y_+$ we have

$$Sp_- < -a\sqrt{c} < Sp_+, \qquad Sp_{3-} < \frac{-1}{a\sqrt{c}} < Sp_{3+}, \qquad \text{and} \qquad Sp_{3\pm} < Sp_\pm < 0,$$

(compare (4.72)). See Figure 4.3, in which we indicate the location of the dominant saddles in the case $a = \frac{5}{6}$, $c = \frac{1}{3}$ and $\beta = \frac{9}{8}$.

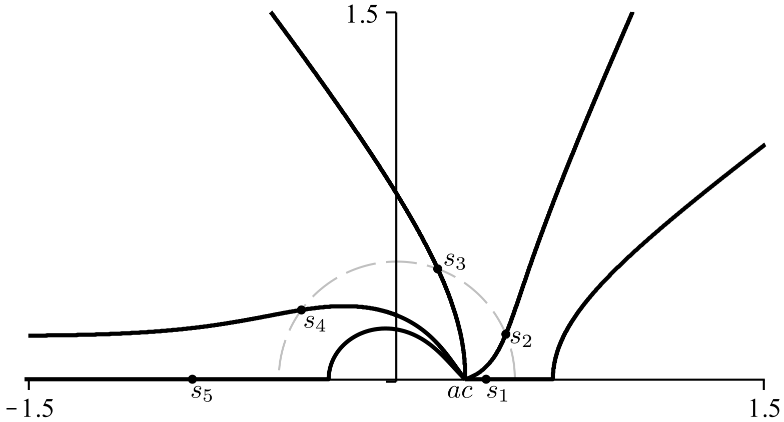

Figure 4.3: Steepest descent contours $\mathcal{C}_+$ in the cases $y = y_j$, where $y_1 = \frac{1}{5} < Y_-$, $y_2 = \frac{2}{5}$, $y_3 = \frac{7}{5}$, $y_4 = \frac{17}{5}$ and $y_5 = 4 > Y_+$. The saddle points are located at s_j. Note that the contours emanate from ac and that s_2, s_3, s_4 are located on the circle $|t| = a\sqrt{c}$.

4.4 The multi-valuedness of the hypergeometric function

The principal branch for the hypergeometric function ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)$ is $|\mathrm{ph}(1-z)| < \pi$. Hence, the branch-cut is $z > 1$. Thus for the functions $G_\pm(t)$ the t-branch-cuts are at points where $1/z(t) > 1$, that is, $0 < z(t) < 1$. The reader can check that this happens on the intervals $(-\infty, 0)$, $(a, 1/a)$ and on the unit circle $|t| = 1$.

For the function $G_3(t)$ the t-branch-cuts are at points where $1 - 1/z(t) > 1$, that is, $z(t) < 0$. This happens on the intervals $(0, ac)$ and $(\frac{1}{ac}, \infty)$.

If we continue to use the principal branches for the hypergeometric functions in the definition for $G_\pm(t)$, then we have for $\Im t > 0$

$$\begin{aligned} G(t) &= G_+(t) + G_3(t), && \text{for } |t| < 1, \\ G(t) &= G_+(t) + e^{2(\beta-\gamma-1)\pi i} G_3(t), && \text{for } |t| > 1. \end{aligned} \tag{4.35}$$

Below we will see that the contributions of $G_3(t)$ are exponentially small compared with the contributions of $G_\pm(t)$. It follows from (4.35) that we can use (4.21) for the hypergeometric functions in the right-hand side of (4.27) along the entire contour of integration.

4.5 The case $Y_- < y < Y_+$

Since

$$\Re\left(f\left(Sp_\pm\right) - f_3\left(Sp_{3\pm}\right)\right) = \ln\left(a^2 c\right) < 0, \tag{4.36}$$

it follows that the $G_\pm(t)$-integrals in (4.30) dominate the asymptotics. For the $G_+(t)$ integral the steepest-descent contour of integration in the upper half t-plane, starts at $t = ac$ passes through the saddle point at $t = Sp_+$ and goes to infinity. (See figure 4.3.) The contour of the $G_-(t)$ integral in the lower half plane is the complex conjugate of the contour in the upper half plane.

Let us study the integral

$$\frac{1}{2\pi i}\int_{\mathcal{C}_+} \frac{\widetilde{G_+}(t)}{t^{n+1}}\,dt, \tag{4.37}$$

where

$$\frac{\widetilde{G_+}(t)}{t^{n+1}} = e^{-x\pi i} K e^{-nf(t)} g(t)/t, \tag{4.38}$$

with K, $g(t)$ and $f(t)$ defined in (4.28), (4.29) and (4.32), respectively. Using the saddle point method (see §2.4(iv) in [34]), the above integral, and hence $S_n(x)$ can be approximated by

$$S_n(x) \sim 2\Re\left(\frac{Ke^{-x\pi i}\left(a\sqrt{c}e^{\theta i}\right)^{-n-\gamma}}{(1-a\sqrt{c}e^{\theta i})\sqrt{2\pi n\left(e^{2\theta i}-1\right)}}\frac{\left(\frac{e^{\theta i}}{\sqrt{c}}-1\right)^{x+\gamma+1/2}}{(1-e^{\theta i}\sqrt{c})^{x+\beta-\gamma-3/2}}\right), \tag{4.39}$$

as $n \to \infty$, where $x = ny$ and θ is defined in (4.34).

The saddle point approximation gives good results: Taking the same a, c, β as before, and $y = 0.7$ then $S_{50}(35) = -8.24876 \times 10^{22}$ and the dominant approximation from the 2 saddle points gives us -8.27800×10^{22}. Hence, only one term gives us already a good approximation.

Regarding the zeros, they can be found using (4.39). Writing the expression in (4.39) as

$$S_n(x) \sim 2\Re(\tilde{K}e^{-\tilde{\theta}i}) = 2\tilde{K}\cos\tilde{\theta}, \tag{4.40}$$

where

$$\begin{aligned}\tilde{\theta} = x\pi + (n+\gamma)\theta - (x+\gamma+1/2)\mathrm{ph}\left(\frac{e^{\theta i}}{\sqrt{c}} - 1\right) + \mathrm{ph}\left(1 - a\sqrt{c}e^{\theta i}\right)\\ +\frac{1}{2}\mathrm{ph}\left(e^{2\theta i} - 1\right) + (x+\beta-\gamma-3/2)\mathrm{ph}\left(1 - e^{\theta i}\sqrt{c}\right)\end{aligned} \tag{4.41}$$

Now by simplifying the above result one can find that zeros are approximately located where

$$n\big(\theta + y(\pi-\theta)\big) + \big(2ny + \beta - 1\big)\,\mathrm{ph}\Big(1 - \sqrt{c}e^{\theta i}\Big) + \mathrm{ph}\Big(1 - a\sqrt{c}e^{\theta i}\Big) = (k+\tfrac{1}{4})\pi, \tag{4.42}$$

where k is an integer. For example, one solution of (4.42) is $y \approx 0.709082395$ that corresponds to $k = 25$. Hence, there should be a zero at $x = 35.45412$. Since we take $x = ny$, so the given value of x is a large zero. The 'exact' zero is located at $x = 35.45469$.

One can note that for fixed n, the polynomials $S_n(x)$ will have finitely many zeros. Hence not every k will produce a zero.

The approximations in this section hold for $Y_- < y < Y_+$, that is, $0 < \theta < \pi$. (Note the factor $e^{2\theta i} - 1$ in (4.39).) The saddle points $Sp_\pm = a\sqrt{c}e^{\pm\theta i}$ coalesce when $\theta = 0, \pi$. In the next sections we will obtain asymptotic approximations that hold in larger intervals, including the turning points at $\theta = 0, \pi$.

4.6 The case $y \approx Y_-$

We write (4.34) as

$$\cos\theta = 1 - \frac{1-c}{2\sqrt{c}}\left(y - Y_-\right), \qquad \text{and take} \quad \begin{array}{lcr} y < Y_- & \Longrightarrow & -\theta i > 0, \\ y > Y_- & \Longrightarrow & \theta > 0. \end{array} \tag{4.43}$$

Hence, if $y \approx Y_-$ then $\theta \approx 0$.

To obtain a uniform asymptotic approximation we will use the cubic transformation suggested by Chester, Friedman and Ursell [6]

$$f(t) = \tfrac{1}{3}u^3 + \omega u + \psi. \tag{4.44}$$

The right-hand side of (4.44) has sadde points at $u = \mp i\sqrt{\omega}$, and we will insist that these correspond to $t = Sp_{\pm} = a\sqrt{c}e^{\pm\theta i}$, respectively. This gives us the following two results

$$\psi = \ln(a) + \tfrac{1}{2}(1+y)\ln(c), \tag{4.45}$$

and

$$-\tfrac{2}{3}i\omega^{3/2} = \theta i(1+y) + y\ln\left(\frac{e^{-\theta i}-\sqrt{c}}{e^{\theta i}-\sqrt{c}}\right). \tag{4.46}$$

The reader can check that for the right-hand side in (4.46), we have

$$\theta i(1+y) + y\ln\left(\frac{e^{-\theta i}-\sqrt{c}}{e^{\theta i}-\sqrt{c}}\right) \sim -\tfrac{2}{3}i\frac{c^{-1/4}\left(1+\sqrt{c}\right)^2}{\sqrt{1-c}}\left(y-Y_-\right)^{3/2}, \tag{4.47}$$

as $y \to Y_-$. Hence,

$$\omega \sim \frac{c^{-1/6}\left(1+\sqrt{c}\right)^{4/3}}{(1-c)^{1/3}}\left(y-Y_-\right), \quad \text{as} \quad y \to Y_-. \tag{4.48}$$

It is not difficult to show that on the interval $y \in (0, Y_+)$, $\omega(y)$ is an increasing analytic function of y with

$$\omega(0) = -\left(\tfrac{3}{4}\ln(1/c)\right)^{2/3}, \quad \omega\left(Y_-\right) = 0, \quad \omega\left(Y_+\right) = \left(\frac{3\pi\sqrt{c}}{1-\sqrt{c}}\right)^{2/3}. \tag{4.49}$$

The local behaviour of transformation (4.44) near $t \approx a\sqrt{c}e^{\theta i}$ and $u \approx -i\sqrt{\omega}$ is

$$\frac{-ie^{-2\theta i}\sin\theta}{a^2\sqrt{c}(1-c)y}\left(t - a\sqrt{c}e^{\theta i}\right)^2 \approx -i\sqrt{\omega}\left(u + i\sqrt{\omega}\right)^2, \tag{4.50}$$

from which it follows that in the case $\omega < 0$ we have to take $\sqrt{\omega} = i\sqrt{|\omega|}$.

Integral (4.37) becomes

$$\frac{1}{2\pi i}\int_{\mathcal{C}_+}\frac{\widetilde{G_+}(t)}{t^{n+1}}\,dt = \frac{e^{-x\pi i}Ke^{-n\psi}}{2\pi i}\int_{\infty e^{-2\pi i/3}}^{\infty} e^{-n(\frac{1}{3}u^3+\omega u)}g_0(u)\,du, \tag{4.51}$$

where

$$g_0(u) = t^{-1}g(t)\frac{dt}{du} = t^{-1}g(t)\frac{u^2+\omega}{f'(t)}. \tag{4.52}$$

The orientation of the u-integral in (4.51) follows when one takes the obvious square-roots in (4.50). Note that from l'Hôpital's rule we obtain

$$\frac{dt}{du}\Big|_{u=\pm i\sqrt{\omega}} = \sqrt{\frac{\pm 2i\sqrt{\omega}}{f''(a\sqrt{c}e^{\mp\theta i})}}. \tag{4.53}$$

It follows that

$$g_0(\pm i\sqrt{\omega}) = \sqrt{\frac{\sqrt{w}(1-c)y}{\sqrt{c}\sin\theta}}\left(\frac{(1-c)y}{ac}\right)^{\gamma}\frac{\left(1-\sqrt{c}e^{\mp\theta i}\right)^{1-\beta}}{1-a\sqrt{c}e^{\mp\theta i}}. \tag{4.54}$$

To obtain a uniform asymptotic approximation, we use Bleistein's Method and substitute into (4.51)

$$g_0(u) = p + qu + (u^2+\omega)h_0(u), \tag{4.55}$$

where

$$p = \frac{g_0(i\sqrt{\omega}) + g_0(-i\sqrt{\omega})}{2}, \qquad q = \frac{g_0(i\sqrt{\omega}) - g_0(-i\sqrt{\omega})}{2i\sqrt{\omega}}, \tag{4.56}$$

and obtain

$$\frac{1}{2\pi i}\int_{\mathcal{C}_+}\frac{\widetilde{G_+}(t)}{t^{n+1}}\,dt \sim \frac{e^{-x\pi i}Ke^{-n\psi}}{2\pi i}\left(p\int_{\infty e^{-2\pi i/3}}^{\infty} e^{-n(\frac{1}{3}u^3+\omega u)}\,du \right.$$
$$\left. +q\int_{\infty e^{-2\pi i/3}}^{\infty} u e^{-n(\frac{1}{3}u^3+\omega u)}\,du\right). \tag{4.57}$$

Using the change of variable $u = e^{-\pi i/3}n^{-1/3}t$, we get

$$\frac{1}{2\pi i}\int_{\mathcal{C}_+}\frac{\widetilde{G_+}(t)}{t^{n+1}}\,dt \sim \frac{e^{-x\pi i}Ke^{-n\psi}}{2\pi i}\left(\frac{pe^{-\pi i/3}}{n^{1/3}}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} e^{\frac{1}{3}t^3-\omega n^{2/3}e^{-\pi i/3}t}\,dt \right.$$
$$\left. +\frac{qe^{-2\pi i/3}}{n^{2/3}}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} t e^{\frac{1}{3}t^3-\omega n^{2/3}e^{-\pi i/3}t}\,dt\right). \tag{4.58}$$

Which is equivalent to

$$\frac{1}{2\pi i}\int_{\mathcal{C}_+}\frac{\widetilde{G_+}(t)}{t^{n+1}}\,dt \sim e^{-x\pi i}Ke^{-n\psi}\left(\frac{pe^{-\pi i/3}}{n^{1/3}}\mathrm{Ai}\left(\omega e^{-\pi i/3}n^{2/3}\right)\right.$$
$$\left. -\frac{qe^{-2\pi i/3}}{n^{2/3}}\mathrm{Ai}'\left(\omega e^{-\pi i/3}n^{2/3}\right)\right), \tag{4.59}$$

as $n\to\infty$, where $\mathrm{Ai}(z)$ is the Airy function, see (9.5.4) in [32].

Since the polynomials are real-valued and the contribution of the $\mathcal{C}_-$ will be just the complex conjugate of (4.59), we conclude that

$$S_n(x) \sim \quad 2\Re\left\{e^{-x\pi i}Ke^{-n\psi}\left(\frac{pe^{-\pi i/3}}{n^{1/3}}\mathrm{Ai}\left(\omega e^{-\pi i/3}n^{2/3}\right)\right.\right. \\ \left.\left.-\frac{qe^{-2\pi i/3}}{n^{2/3}}\mathrm{Ai}'\left(\omega e^{-\pi i/3}n^{2/3}\right)\right)\right\}, \tag{4.60}$$

as $n \to \infty$, uniformly for $y \in [\varepsilon, Y_+ - \varepsilon]$, where ε is a small positive constant.

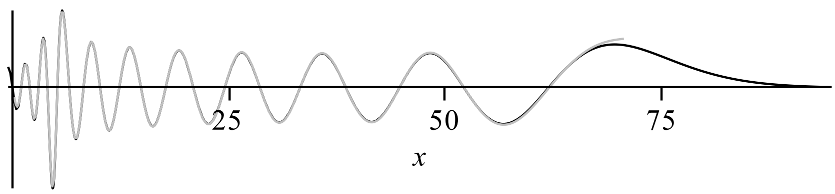

Figure 4.4: The graph of a rescaled version of $S_{20}(x)$, where $a = \frac{5}{6}$, $c = \frac{1}{3}$, $\beta = \frac{9}{8}$ (black), and approximation (4.60) (grey). Note that only near $y = 0$ and $y = Y_+$ the difference is visible.

4.7 The case $0 \leqslant y < Y_-$

We will use the notation of the previous section. In §4.2 we dealt with the case of large n and finite x, that is, $y \approx 0$, and in the previous section we covered the case $0 < y < Y_+$. In the case of $0 \leqslant y < Y_-$ the active saddle point of phase function $f(t)$ is at $t = Sp_+$, and we have $ac \leqslant Sp_+ < a\sqrt{c}$. In integral representation (4.30) the main integrals are still the ones involving $G_\pm$. We write

$$\frac{1}{2\pi i}\int_{\mathcal{C}_\pm}\frac{G_\pm(t)}{t^{n+1}}\,dt = \frac{e^{\mp x\pi i}K}{2\pi i}\int_{\mathcal{C}_\pm}e^{-nf(t)}\tilde{g}(t)\,dt, \tag{4.61}$$

where

$$\tilde{g}(t) = \frac{t^{-\gamma-1}\left(1-\frac{t}{a}\right)^{\gamma-\beta+1}}{(1-t)\left(\frac{t}{ac}-1\right)^{-\gamma}}\,{}_2F_1\left(\begin{matrix}\gamma, 2+\gamma-\beta\\ 1+\gamma+x\end{matrix};\frac{1}{z(t)}\right) \tag{4.62}$$

Since x appears in the bottom entry of the hypergeometric function possible large values for x has only a positive influence in the expansion of this function near $t = Sp_+$.

Note that as $y \to 0$ the saddle point of phase function $f(t)$ at $t = Sp_+$ coalesces with the branch-point at $t = ac$. The local behaviour of $f(t)$ near $t = ac$ is

$$f(t) \sim -y \ln\left(\frac{t}{ac} - 1\right) + \ln(ac) + y\ln(1-c), \qquad \text{as} \ \ t \to ac. \tag{4.63}$$

For these reasons we choose the transformation

$$f(t) = u - \alpha \ln(u) + \ln(ac) + y \ln(1-c). \tag{4.64}$$

The right-hand side has a saddle point at $u = \alpha$ and a branch-point at $u = 0$. Furthermore

$$\frac{t}{ac} - 1 \sim u^{\alpha/y}, \qquad \text{as} \ \ u \to 0. \tag{4.65}$$

We insist that the saddle point at $t = Sp_+$ is mapped to $u = \alpha$. Hence, α is defined via

$$f\left(Sp_+\right) = \alpha(1 - \ln\alpha) + \ln(ac) + y\ln(1-c), \tag{4.66}$$

and the property that $\alpha \sim y$ as $y \to 0$. Since

$$\frac{dt}{du} = \frac{u-\alpha}{uf'(t)} = \frac{t(t-a)(t-ac)(u-\alpha)}{(t-Sp_+)(t-Sp_-)u}, \tag{4.67}$$

it follows that

$$\frac{dt}{du}_{|u=\alpha} = \frac{1}{\sqrt{\alpha f''(Sp_+)}}. \tag{4.68}$$

We have

$$\frac{1}{2\pi i}\int_{\mathcal{C}_\pm} \frac{G_\pm(t)}{t^{n+1}}\, dt = \frac{e^{\mp x\pi i}K}{(ac)^n\,(1-c)^{ny}\,2\pi i}\int_0^\infty e^{-n(u-\alpha\ln u)}\tilde{g}_0(u)u^{\frac{\alpha}{y}(\gamma+1)-1}\, du, \tag{4.69}$$

where

$$\tilde{g}_0(u) = \tilde{g}(t)\frac{dt}{du}u^{1-\frac{\alpha}{y}(\gamma+1)}. \tag{4.70}$$

The power of u in (4.70) is chosen such that $\tilde{g}_0(u)$ has no branch-point at $u = 0$.

To obtain a uniform asymptotic expansion we have to expand $\tilde{g}_0(u)$ near $u = \alpha$. Hence,

$$S_n(x) \sim \frac{-\sin(x\pi)K\tilde{g}_0(\alpha)}{(ac)^n\,(1-c)^{ny}\,\pi}n^{-\alpha n-\frac{\alpha}{y}(\gamma+1)}\Gamma\left(\alpha n + \frac{\alpha}{y}(\gamma+1)\right), \tag{4.71}$$

as $n \to \infty$, uniformly for $y \in [0, Y_- - \varepsilon]$, where ε is a small positive constant. Again we can see that $S_n(x)$ has zeros at approximately $x = m$, where m is a bounded nonnegative integer. This is in agreement with the final paragraph of §4.2.

4.8 The case $y \approx Y_+$

When $y \approx Y_+$ we have for the θ in (4.34) $\theta \approx \pi$. We will replace θ by $\pi - \varphi$ and the link between the new φ and y is now

$$\cos\varphi = 1 + \frac{1-c}{2\sqrt{c}}\left(y - Y_+\right), \qquad \text{and take} \quad \begin{array}{ll} y > Y_+ \implies & -\varphi i > 0, \\ y < Y_+ \implies & \varphi > 0. \end{array} \tag{4.72}$$

Hence, if $y \approx Y_+$ then $\varphi \approx 0$.

In this case we use the cubic transformation

$$f(t) = \tfrac{1}{3}u^3 - \omega u + \psi. \tag{4.73}$$

The right-hand side of (4.73) has sadde points at $u = \mp\sqrt{\omega}$, and we will insist that these correspond to $t = Sp_\pm = a\sqrt{c}e^{(\pi\pm\varphi)i}$, respectively. This gives us

$$\psi = \ln(a) + \tfrac{1}{2}(1+y)\ln(c) + i\pi(1-y), \tag{4.74}$$

and

$$\tfrac{2}{3}\omega^{3/2} = (1+y)\varphi i + y\ln\left(\frac{e^{-\varphi i}+\sqrt{c}}{e^{\varphi i}+\sqrt{c}}\right). \tag{4.75}$$

The reader can check that for the right-hand side in (4.75), we have

$$(1+y)\varphi i + y\ln\left(\frac{e^{-\varphi i}+\sqrt{c}}{e^{\varphi i}+\sqrt{c}}\right) \sim \tfrac{2}{3}\frac{c^{-1/4}\left(1-\sqrt{c}\right)^2}{\sqrt{1-c}}\left(y - Y_+\right)^{3/2}, \tag{4.76}$$

as $y \to Y_+$. Hence,

$$\omega \sim \frac{c^{-1/6}\left(1-\sqrt{c}\right)^{4/3}}{\left(1-c\right)^{1/3}}\left(y - Y_+\right), \quad \text{as} \quad y \to Y_+. \tag{4.77}$$

It is not difficult to show that on the interval $y \in (Y_-, \infty)$, $\omega(y)$ is an increasing analytic function of y with

$$\omega\left(Y_-\right) = -\left(\frac{3\pi\sqrt{c}}{1+\sqrt{c}}\right)^{2/3}, \qquad \omega\left(Y_+\right) = 0. \tag{4.78}$$

The local behaviour of transformation (4.73) near $t \approx a\sqrt{c}e^{(\pi-\varphi)i}$ and $u \approx \sqrt{\omega}$ is

$$\frac{-ie^{2\varphi i}\sin\varphi}{a^2\sqrt{c}(1-c)y}\left(t - a\sqrt{c}e^{(\pi-\varphi)i}\right)^2 \approx \sqrt{\omega}\left(u-\sqrt{\omega}\right)^2, \tag{4.79}$$

from which it follows that in the case $\omega < 0$ we have to take $\sqrt{\omega} = -i\sqrt{|\omega|}$.

Using substitution (4.73) integral (4.37) becomes

$$\frac{1}{2\pi i}\int_{\mathcal{C}_+}\frac{\widetilde{G_+}(t)}{t^{n+1}}\,dt = \frac{e^{-x\pi i}Ke^{-n\psi}}{2\pi i}\int_{\infty}^{\infty e^{-2\pi i/3}} e^{-n(\frac{1}{3}u^3-\omega u)}g_0(u)\,du, \tag{4.80}$$

where

$$g_0(u) = t^{-1}g(t)\frac{dt}{du} = t^{-1}g(t)\frac{u^2-\omega}{f'(t)}. \tag{4.81}$$

The orientation of the u-integral in (4.80) follows when one takes the obvious square-roots in (4.79). Note that from l'Hôpital's rule we obtain

$$\frac{dt}{du}_{|u=\pm\sqrt{\omega}} = \sqrt{\frac{\pm 2\sqrt{\omega}}{f''(a\sqrt{c}e^{(\pi\mp\varphi)i})}}. \tag{4.82}$$

It follows that

$$g_0(\pm\sqrt{\omega}) = -\sqrt{\frac{i\sqrt{w}(1-c)y}{\sqrt{c}\sin\varphi}}\left(\frac{(1-c)y}{ac}\right)^{\gamma}\frac{\left(1+\sqrt{c}e^{\mp\varphi i}\right)^{1-\beta}}{1+a\sqrt{c}e^{\mp\varphi i}}. \tag{4.83}$$

Again, we substitute into (4.80)

$$g_0(u) = p + qu + (u^2-\omega)h_0(u), \tag{4.84}$$

where

$$p = \frac{g_0(\sqrt{\omega}) + g_0(-\sqrt{\omega})}{2}, \qquad q = \frac{g_0(\sqrt{\omega}) - g_0(-\sqrt{\omega})}{2\sqrt{\omega}}, \tag{4.85}$$

and obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathcal{C}_+}\frac{\widetilde{G_+}(t)}{t^{n+1}}\,dt \sim \frac{-e^{-x\pi i}Ke^{-n\psi}}{2\pi i}&\left(p\int_{\infty e^{-2\pi i/3}}^{\infty} e^{-n(\frac{1}{3}u^3-wu)}\,du\right.\\ &\left.+q\int_{\infty e^{-2\pi i/3}}^{\infty} ue^{-n(\frac{1}{3}u^3-wu)}\,du\right).\end{aligned} \tag{4.86}$$

Note that in this case $e^{-x\pi i}e^{-n\psi} = (-a)^{-n}c^{-(n+x)/2}$. Using again the change of variable

$u = e^{-\pi i/3} n^{-1/3} t$, we get

$$\frac{1}{2\pi i}\int_{\mathcal{C}_+} \frac{\widetilde{G_+}(t)}{t^{n+1}}\,dt \sim \frac{K(-a)^{-n}}{2\pi i c^{(n+x)/2}}\left(\frac{pe^{2\pi i/3}}{n^{1/3}}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} e^{\frac{1}{3}t^3 - wn^{2/3}e^{2\pi i/3}t}\,dt + \frac{qe^{\pi i/3}}{n^{2/3}}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} te^{\frac{1}{3}t^3 - wn^{2/3}e^{2\pi i/3}t}\,dt\right), \quad (4.87)$$

which is equivalent to,

$$\frac{1}{2\pi i}\int_{\mathcal{C}_+} \frac{\widetilde{G_+}(t)}{t^{n+1}}\,dt \sim \frac{K(-a)^{-n}}{c^{(n+x)/2}}\left(\frac{pe^{2\pi i/3}}{n^{1/3}}\mathrm{Ai}\left(\omega e^{2\pi i/3}n^{2/3}\right) - \frac{qe^{\pi i/3}}{n^{2/3}}\mathrm{Ai}'\left(\omega e^{2\pi i/3}n^{2/3}\right)\right), \quad (4.88)$$

as $n \to \infty$. Hence,

$$S_n(x) \sim \frac{2K(-a)^{-n}}{c^{(n+x)/2}}\Re\left\{\frac{pe^{2\pi i/3}}{n^{1/3}}\mathrm{Ai}\left(\omega e^{2\pi i/3}n^{2/3}\right) - \frac{qe^{\pi i/3}}{n^{2/3}}\mathrm{Ai}'\left(\omega e^{2\pi i/3}n^{2/3}\right)\right\}, \quad (4.89)$$

as $n \to \infty$. Using the fact that p and q are real-valued and the connection relation

$$\mathrm{Ai}(\omega) + e^{-2\pi i/3}\mathrm{Ai}\left(\omega e^{-2\pi i/3}\right) + e^{2\pi i/3}\mathrm{Ai}\left(\omega e^{2\pi i/3}\right) = 0, \quad (4.90)$$

(see (9.2.12) in [32]), we get

$$S_n(x) \sim \frac{K(-a)^{-n}}{c^{(n+x)/2}}\left\{\frac{-p}{n^{1/3}}\mathrm{Ai}\left(\omega n^{2/3}\right) - \frac{q}{n^{2/3}}\mathrm{Ai}'\left(\omega n^{2/3}\right)\right\}, \quad (4.91)$$

as $n \to \infty$, uniformly for $y \in [Y_- + \varepsilon, Y_+ + 1/\varepsilon]$, where ε is a small positive constant.

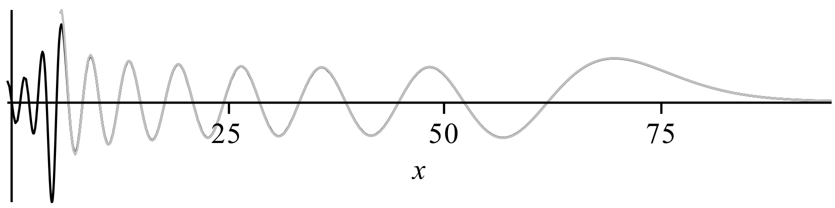

Figure 4.5: The graph of a rescaled version of $S_{20}(x)$, where $a = \frac{5}{6}$, $c = \frac{1}{3}$, $\beta = \frac{9}{8}$ (black), and approximation (4.91) (grey). Note that only near $y = Y_-$ the difference is visible.

4.9 Large zeros

Let a_k be the k^{th} zero of the Airy function $\mathrm{Ai}(x)$ with $k = 1, 2, 3, \cdots$. Thus $a_1 = -2.338\cdots$, $a_2 = -4.088\cdots$. (See §9.9 in [32].) The dominant term in (4.91) includes the factor $\mathrm{Ai}\left(\omega n^{2/3}\right)$. Its zeros are located at

$$\omega_k = a_k n^{-2/3}. \tag{4.92}$$

Hence, for fixed k these will approach zero as $n \to \infty$, and we can use (4.77)

$$a_k n^{-2/3} \sim \frac{1-\sqrt{c}}{(c+\sqrt{c})^{1/3}} \left(y_{n,k} - Y_+\right), \tag{4.93}$$

as $n \to \infty$, where $x = n y_{n,j}$ are the zeros of $S_n(x)$ with

$$0 < y_{n,n} < y_{n,n-1} < \ldots < y_{n,1} < \infty. \tag{4.94}$$

Thus for the large zeros we obtain,

$$y_{n,k} \sim Y_+ + \frac{(c+\sqrt{c})^{1/3}}{1-\sqrt{c}} \frac{a_k}{n^{2/3}}, \tag{4.95}$$

as $n \to \infty$, where $k = 1, 2, 3, \cdots$ is fixed.

To obtain a better approximation for the zeros we use both terms in the right-hand side of (4.91). Its small zeros are located at $\omega_k = a_k n^{-2/3} + \delta$, where $\delta = o\left(n^{-2/3}\right)$, as $n \to \infty$. We substitute this expression for ω into the right-hand side of (4.91), approximate $\mathrm{Ai}\left(\omega_k n^{2/3}\right) \approx \delta n^{2/3} \mathrm{Ai}'(a_k)$ and $\mathrm{Ai}'\left(\omega_k n^{2/3}\right) \approx \mathrm{Ai}'(a_k)$. Hence, $\delta \approx -q/(pn)$. From (4.85), (4.83) and (4.75) we can obtain the limit of q/p as $\omega \to 0$. The result is the approximation

$$\omega_k \sim a_k n^{-2/3} + \left(\frac{a}{1+a\sqrt{c}} - \frac{1-\beta}{1+\sqrt{c}}\right) \frac{(c+\sqrt{c})^{2/3}}{n}, \qquad \text{as } n \to \infty. \tag{4.96}$$

Hence,

$$y_{n,k} \sim Y_+ + \frac{(c+\sqrt{c})^{1/3}}{1-\sqrt{c}} \frac{a_k}{n^{2/3}} + \left(\frac{a}{1+a\sqrt{c}} - \frac{1-\beta}{1+\sqrt{c}}\right) \frac{c+\sqrt{c}}{(1-\sqrt{c})\, n}, \tag{4.97}$$

as $n \to \infty$, where $k = 1, 2, 3, \cdots$ is fixed.

When we let $\lambda \to 0$, that is, $a \to 1$, we obtain a two term approximation for the large zeros of the classical Meixner polynomials:

$$y_{n,k} \sim Y_+ + \frac{(c+\sqrt{c})^{1/3}}{1-\sqrt{c}} \frac{a_k}{n^{2/3}} + \frac{\beta\sqrt{c}}{(1-\sqrt{c})\,n}, \tag{4.98}$$

as $n \to \infty$, where $k = 1, 2, 3, \cdots$ is fixed. The first two terms in this approximation agree with the result (2.42) given in [16]. The third term appears to be a new term, and is surprisingly simple.

We finish with a numerical illustration. Taking $n = 100$ and $k = 1$ in (4.95) we obtain $y_{n,1} \approx 3.4831614$, and from (4.97) we obtain $y_{n,1} \approx 3.4969920$. The 'exact' location is $y_{n,1} = 3.4999640$. The errors seem to be of the correct order.

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Chapter 5

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